Number and Algebra

IB HL Study Guide

Contents

IB Math AA HL — Number & Algebra

Complete Study Guide

Topics Covered

1. Sequences & Series — arithmetic and geometric, sigma notation, infinite sums
2. Exponents & Logarithms — laws, change of base, natural log, solving equations
3. Binomial Theorem — expansion, general term, finding specific terms
4. Proof by Induction HL — sum formulas, divisibility, formal structure
5. Systems of Equations HL — 3×3 systems, Gaussian elimination, geometric interpretation
6. Practice Questions & Exam Alerts

Topic 1 of the IB Math AA HL syllabus — Paper 1 and Paper 2

IB TIP

Formula booklet awareness: The arithmetic and geometric sequence and series formulas, the binomial theorem, and the general binomial term formula are all in the IB formula booklet. However, the booklet does **not** provide the proof by induction structure or criteria for infinite geometric series convergence. Know those cold.

MEMORISE THIS

Core formulas at a glance

Formula	Expression
Arithmetic nth term	$u_n = u_1 + (n-1)d$
Arithmetic sum	$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$
Geometric nth term	$u_n = u_1 \cdot r^{n-1}$
Geometric sum	$S_n = \frac{u_1(1-r^n)}{1-r},\ r \neq 1$
Infinite geometric sum	$S_\infty = \frac{u_1}{1-r},\ \lvert r\rvert < 1$
Binomial general term	$T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Section 1: Sequences & Series

A **sequence** is an ordered list of numbers following a rule. A **series** is the sum of the terms of a sequence. On the IB exam, sequences and series questions appear on both Paper 1 (no calculator) and Paper 2, and they frequently combine with logarithms or proofs in multi-part questions.

1.1 Arithmetic Sequences & Series

In an **arithmetic sequence**, each term is obtained by adding a fixed number d (the **common difference**) to the previous term.

$$u_n = u_1 + (n-1)d$$

The sum of the first n terms of an arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \qquad \text{or equivalently} \qquad S_n = \frac{n}{2}(u_1 + u_n)$$

The second form is often faster when you already know the first and last terms.

✏ **WORKED EXAMPLE**

Arithmetic sequence — finding a term and sum

The fourth term of an arithmetic sequence is 13 and the tenth term is 37. Find the common difference, the first term, and S_{20}.

Step 1: Set up simultaneous equations using $u_n = u_1 + (n-1)d$:

$$u_4 = u_1 + 3d = 13 \; u_{10} = u_1 + 9d = 37$$

Step 2: Subtract the first equation from the second:

$$6d = 24 \implies d = 4$$

Step 3: Substitute back to find u_1:

$$u_1 + 3(4) = 13 \implies u_1 = 1$$

Step 4: Compute S_{20}:

$$S_{20} = \frac{20}{2}\big(2(1) + 19(4)\big) = 10(2+76) = 10 \times 78 = 780$$

⚠ **EXAM ALERT**

Common trap — arithmetic series: Students often substitute $n-1$ instead of n when identifying the number of terms in a sum. For example, the sum from u_3 to u_{10} contains $10 - 3 + 1 = 8$ terms, not 7. Always count terms explicitly before using S_n.

1.2 Geometric Sequences & Series

In a **geometric sequence**, each term is obtained by multiplying the previous term by a fixed number r (the **common ratio**):

$$u_n = u_1 \cdot r^{n-1}$$

Sum of the first n terms (for $r \neq 1$):

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

An equivalent form sometimes used when $r > 1$:

$$S_n = \frac{u_1(r^n - 1)}{r - 1}$$

Infinite geometric series: When $|r| < 1$, the partial sums converge:

$$S_\infty = \frac{u_1}{1-r}, \qquad |r| < 1$$

⚠ **EXAM ALERT**

Critical condition: You **must** state $|r| < 1$ whenever you use S_∞. An answer that uses the infinite sum formula without verifying or stating the convergence condition will lose marks in any IB mark scheme. If $|r| \geq 1$, the infinite sum does not exist.

WORKED EXAMPLE

Geometric series — infinite sum and finding r

A geometric series has first term 12 and sum to infinity 20. Find the common ratio and the sum of the first 5 terms.

Step 1: Use $S_\infty = \dfrac{u_1}{1 - r}$:

$$20 = \frac{12}{1-r} \implies 1 - r = \frac{12}{20} = 0.6 \implies r = 0.4$$

Step 2: Verify convergence: $|0.4| < 1$ ✓

Step 3: Find S_5:

$$S_5 = \frac{12(1-0.4^5)}{1-0.4} = \frac{12(1-0.01024)}{0.6} = \frac{12\times 0.98976}{0.6} = \frac{11.877\,12}{0.6} \approx 19.80$$

1.3 Sigma Notation

Sigma notation provides a compact way to write series:

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n$$

Key properties:

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \qquad \sum_{k=1}^{n} c \cdot a_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

IB TIP

Reading sigma notation on Paper 1: Always identify whether the index starts at 0 or 1 — it changes the number of terms. Also check whether the expression inside the sigma is arithmetic (linear in k) or geometric (exponential in k) to select the right formula.

Section 2: Exponents & Logarithms

2.1 Laws of Exponents

For $a > 0, b > 0$, and $m, n \in \mathbb{R}$:

Law	Expression
Product	$a^m \cdot a^n = a^{m+n}$
Quotient	$\frac{a^m}{a^n} = a^{m-n}$
Power of power	$(a^m)^n = a^{mn}$
Negative exponent	$a^{-n} = \frac{1}{a^n}$
Fractional exponent	$a^{m/n} = \sqrt[n]{a^m}$

2.2 Laws of Logarithms

For $a > 0, a \neq 1, x > 0, y > 0$:

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^n) = n \log_a x$$

$$\log_a a = 1 \qquad \log_a 1 = 0$$

Change of base:

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad (\text{commonly: } \log_a x = \frac{\ln x}{\ln a})$$

The **natural logarithm** $\ln x = \log_e x$ where $e \approx 2.718$. On IB exams, ln and e^x are inverse functions: $e^{\ln x} = x$ and $\ln(e^x) = x$.

WORKED EXAMPLE

Solving an exponential equation

Solve $3^{2x+1} = 5^x$*, giving your answer in exact form.*

Step 1: Take the natural log of both sides:

$\ln(3^{2x+1}) = \ln(5^x)$

Step 2: Apply the power law:

$(2x+1)\ln 3 = x\ln 5$

Step 3: Expand and collect x terms:

$2x\ln 3 + \ln 3 = x\ln 5 \ln 3 = x\ln 5 - 2x\ln 3 = x(\ln 5 - 2\ln 3)$

Step 4: Solve for x:

$x = \frac{\ln 3}{\ln 5 - 2\ln 3} = \frac{\ln 3}{\ln 5 - \ln 9} = \frac{\ln 3}{\ln(5/9)}$

EXAM ALERT

Log of a negative number: $\ln x$ and $\log_a x$ are only defined for $x > 0$. After solving a logarithmic equation, always check that any proposed solution does not require taking the log of a non-positive number. Reject such extraneous solutions and state why.

MEMORISE THIS

Inverse relationships to memorise:

$a^{\log_a x} = x \qquad \log_a(a^x) = x$

$e^{\ln x} = x \qquad \ln(e^x) = x$

These appear regularly in simplification and equation-solving steps.

Section 3: Binomial Theorem

3.1 The Expansion

The **binomial theorem** states that for $n \in \mathbb{Z}^+$:

$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

where the **binomial coefficient** is:

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Pascal's triangle provides $\binom{n}{r}$ values for small n, but for larger n you should use the formula directly or the GDC.

3.2 The General Term

The $(r+1)$th term (also written T_{r+1}) of the expansion of $(a+b)^n$ is:

$T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Note the indexing: r starts at 0, so the **first term** is T_1 (with $r = 0$) and the **last term** is T_{n+1} (with $r = n$).

IB TIP

The most common binomial exam question: "Find the term containing x^k" or "find the coefficient of x^k." The method is always: write the general term T_{r+1}, set the power of x equal to k, solve for r, then substitute back. Never expand the whole binomial if only one term is needed.

WORKED EXAMPLE

Finding a specific term in a binomial expansion

Find the term containing x^3 in the expansion of $\left(2x - \dfrac{1}{x}\right)^9$.

Step 1: Write the general term with $a = 2x$ and $b = -\dfrac{1}{x}$:

$T_{r+1} = \binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x}\right)^r = \binom{9}{r}2^{9-r}(-1)^r \cdot x^{9-r} \cdot x^{-r}$

Step 2: Combine powers of x:

$T_{r+1} = \binom{9}{r}2^{9-r}(-1)^r \cdot x^{9-2r}$

Step 3: Set the power of x equal to 3:

$9 - 2r = 3 \implies r = 3$

Step 4: Substitute $r = 3$:

$T_4 = \binom{9}{3}2^6(-1)^3 \cdot x^3 = 84 \times 64 \times (-1) \cdot x^3 = -5376\,x^3$

Answer: The term containing x^3 is $-5376x^3$.

WORKED EXAMPLE

Binomial expansion with constant term

Find the constant term in the expansion of $\left(x^2 + \dfrac{3}{x}\right)^6$.

Step 1: General term with $a = x^2$ and $b = \dfrac{3}{x}$:

$T_{r+1} = \binom{6}{r}(x^2)^{6-r}\left(\frac{3}{x}\right)^r = \binom{6}{r}3^r \cdot x^{12-2r} \cdot x^{-r} = \binom{6}{r}3^r \cdot x^{12-3r}$

Step 2: For a constant term, set power of x to zero:

$12 - 3r = 0 \implies r = 4$

Step 3: Substitute $r = 4$:

$T_5 = \binom{6}{4}3^4 = 15 \times 81 = 1215$

Answer: The constant term is 1215.

EXAM ALERT

Sign errors in binomial expansions: When b is negative (e.g., $b = -\frac{1}{x}$), the sign alternates with r. Write $(-1)^r$ explicitly in the general term; do not try to track signs mentally. One missed negative sign will cost the entire answer mark.

Section 4: Proof by Induction HL

Proof by mathematical induction is an **HL-only** technique and is consistently tested on Paper 1. It proves that a statement $P(n)$ holds for all integers $n \geq n_0$ (usually $n_0 = 1$).

4.1 The Formal Structure

Every induction proof must contain all four components:

1. **Base case:** Show $P(n_0)$ is true by direct calculation.
2. **Inductive hypothesis:** State "Assume $P(k)$ is true for some $k \geq n_0$", then write out explicitly what that assumption says.
3. **Inductive step:** Using the hypothesis, prove that $P(k+1)$ is true.
4. **Conclusion:** State "Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq n_0$."

EXAM ALERT

The most penalised induction error: Students write "Assume true for $n = k$" without saying what "true" means — i.e., without writing the actual formula for $P(k)$. IB mark schemes require the assumption to be written explicitly. If you omit the explicit statement of $P(k)$, you will lose the mark for the inductive hypothesis even if your algebra is correct.

MEMORISE THIS

Induction checklist for every proof:

1. State $P(n)$ (the claim) clearly at the top.
2. Base case: compute LHS and RHS separately, confirm they are equal (or the divisibility holds).
3. Write: "Assume $P(k)$ is true, i.e., [write the formula/statement for k]."
4. Show $P(k+1)$ follows. In the working, highlight exactly where the inductive hypothesis is substituted.
5. Write the conclusion in full — do not abbreviate it.

4.2 Worked Example: Sum of a Series

WORKED EXAMPLE

Prove by induction that $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ **for all** $n \in \mathbb{Z}^+$**.**

State $P(n)$**:** $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Base case $(n = 1)$**:**

$\text{LHS} = 1$. $\text{RHS} = \dfrac{1 \times 2}{2} = 1$.

$\text{LHS} = \text{RHS}$, so $P(1)$ is true. ✓

Inductive hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.,

$\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$

Inductive step: We need to show $P(k+1)$ is true, i.e., that:

$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$

Starting from the LHS of $P(k+1)$:

$\sum_{r=1}^{k+1} r = \underbrace{\displaystyle\sum_{r=1}^{k} r}_{\text{use hypothesis}} + (k+1) = \frac{k(k+1)}{2} + (k+1)$

Factor out $(k+1)$:

$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1) \cdot \frac{k+2}{2} = \frac{(k+1)(k+2)}{2}$

This is the RHS of $P(k+1)$. ✓

Conclusion: Since $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{Z}^+$. ■

4.3 Worked Example: Divisibility Proof

WORKED EXAMPLE

Prove by induction that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{Z}^+$.

State $P(n)$: $3 \mid (4^n - 1)$, i.e., $4^n - 1 = 3m$ for some integer m.

Base case ($n = 1$):

$4^1 - 1 = 3 = 3 \times 1$. This is divisible by 3. $P(1)$ is true. ✓

Inductive hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.,

$$4^k - 1 = 3m \quad \text{for some integer } m, \text{ so} \quad 4^k = 3m + 1$$

Inductive step: We need to show $P(k+1)$ is true, i.e., $3 \mid (4^{k+1} - 1)$.

$$4^{k+1} - 1 = 4 \cdot 4^k - 1$$

Substitute the inductive hypothesis ($4^k = 3m + 1$):

$$= 4(3m + 1) - 1 = 12m + 4 - 1 = 12m + 3 = 3(4m + 1)$$

Since $4m + 1$ is an integer, $4^{k+1} - 1$ is divisible by 3. $P(k+1)$ is true. ✓

Conclusion: Since $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$, by the principle of mathematical induction $4^n - 1$ is divisible by 3 for all $n \in \mathbb{Z}^+$. ■

IB TIP

Divisibility proof strategy: Isolate 4^k (or the base raised to the k) so you can substitute the inductive hypothesis directly. Then factorise so that the required divisor appears explicitly as a factor. Never skip the factoring step — writing "$\ldots = 3(\ldots)$" is the payoff line that earns the mark.

Section 5: Systems of Equations HL

5.1 Setting Up a 3×3 System

A system of three linear equations in three unknowns x, y, z can be written in **augmented matrix** form:

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

5.2 Gaussian Elimination

The goal is to reduce the augmented matrix to **row echelon form** (upper triangular) using three **elementary row operations** (EROs):

Operation	Notation	Description
Swap rows	$R_i \leftrightarrow R_j$	Interchange two rows
Scale a row	$kR_i \to R_i$	Multiply a row by a non-zero constant
Add multiple	$R_i + kR_j \to R_i$	Add a multiple of one row to another

Once in row echelon form, solve by **back substitution** from the bottom row up.

WORKED EXAMPLE

Gaussian elimination — inconsistent system (no solution)

Solve:

$x + 2y - z = 4$ $2x - y + 3z = 1$ $3x + y + 2z = 11$

Step 1: Write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 1 \\ 3 & 1 & 2 & 11 \end{array}\right)$$

Step 2: Eliminate x from rows 2 and 3.

$R_2 \to R_2 - 2R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 3 & 1 & 2 & 11 \end{array}\right)$$

$R_3 \to R_3 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & -5 & 5 & -1 \end{array}\right)$$

Step 3: Eliminate y from row 3.

$R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

Step 4: The last row reads $0 = 6$, a **contradiction**. This system has **no solution** (the planes are inconsistent).

IB TIP

Adjusting the constants for unique vs. no vs. infinite solutions: In exam questions, the system often has a parameter (e.g., k) in one entry. The three cases arise from the bottom row of the reduced matrix: if it gives $0 = c$ with $c \neq 0$, no solution; if $0 = 0$, infinitely many solutions; if the diagonal is non-zero throughout, a unique solution exists.

5.3 Geometric Interpretation

Each equation $ax + by + cz = d$ represents a **plane** in three-dimensional space. A 3×3 system describes the intersection of three planes:

Reduced form outcome	Geometric meaning	Number of solutions
Full diagonal (no zeros)	Three planes meet at a single point	Unique solution
Row of zeros, consistent	Planes share a line or a plane	Infinitely many
Row $0\,0\,0 \mid c, c \neq 0$	Planes are inconsistent	No solution

MEMORISE THIS

Recognising solution type from the augmented matrix:

- Last row all zeros $\to \infty$ solutions (introduce a parameter t)
- Last row $0\ 0\ 0 \mid c$ with $c \neq 0 \to$ no solution
- All pivots non-zero $\to$ unique solution

Always state which case applies — IB mark schemes award a mark for the correct interpretation.

Section 6: Practice Questions

Question 1 — Arithmetic series (Paper 1 style)

The sum of the first n terms of an arithmetic series is $S_n = 3n^2 + 2n$.

(a) Find the first term and the common difference.

(b) Find the 15th term.

▶ Show full solution

Question 2 — Infinite geometric series

A geometric series has $u_1 = 8$ and $u_3 = 2$.

(a) Find the two possible values of the common ratio.

(b) For the value of r for which the series converges, find S_∞.

(c) Find the smallest value of n such that $S_n > 0.99\, S_\infty$.

▶ Show full solution

Question 3 — Binomial theorem

(a) Expand $(1 + 2x)^5$ in ascending powers of x up to and including the term in x^3.

(b) Find the coefficient of x^4 in the expansion of $(3 - x)^7$.

(c) HL Find the value of k such that the expansion of $(k+x)^6$ has a term in x^2 equal to $60x^2$.

▶ Show full solution

Question 4 — Proof by induction HL

Prove by mathematical induction that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{Z}^+$.

▶ Show full solution

Question 5 — Systems of equations HL

Consider the system of equations:

$x + 2y + z = 3$ $2x + y + az = b$ $x - y + 2z = 1$

(a) Show that when $a = 3$ and $b = 4$, the system has infinitely many solutions.

(b) Find the condition on a and b under which the system has a unique solution.

▶ Show full solution

Question 6 — Mixed: logarithms and series

The nth term of a sequence is $u_n = \log_2(2^n \cdot 3)$. Show that the sequence is arithmetic and find its common difference.

▶ Show full solution

⚠ **EXAM ALERT**

PDF download: A print-ready version of this guide with all formulas and worked examples is available at the link below.

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