Math AA HL

IB HL — Question Bank

Q1.

Arithmetic sequence — finding a term and sum

The fourth term of an arithmetic sequence is 13 and the tenth term is 37. Find the common difference, the first term, and S_{20}.

Step 1: Set up simultaneous equations using $u_n = u_1 + (n-1)d$:

$u_4 = u_1 + 3d = 13$ $u_{10} = u_1 + 9d = 37$

Step 2: Subtract the first equation from the second:

$6d = 24 \implies d = 4$

Step 3: Substitute back to find u_1:

$u_1 + 3(4) = 13 \implies u_1 = 1$

Step 4: Compute S_{20}:

$S_{20} = \frac{20}{2}\big(2(1) + 19(4)\big) = 10(2 + 76) = 10 \times 78 = 780$

Q2.

Geometric series — infinite sum and finding r

A geometric series has first term 12 and sum to infinity 20. Find the common ratio and the sum of the first 5 terms.

Step 1: Use $S_\infty = \dfrac{u_1}{1-r}$:

$20 = \frac{12}{1-r} \implies 1 - r = \frac{12}{20} = 0.6 \implies r = 0.4$

Step 2: Verify convergence: $|0.4| < 1$ ✓

Step 3: Find S_5:

$S_5 = \frac{12(1-0.4^5)}{1-0.4} = \frac{12(1-0.01024)}{0.6} = \frac{12\times 0.98976}{0.6} = \frac{11.877\,12}{0.6} \approx 19.80$

Q3.

Solving an exponential equation

Solve $3^{2x+1} = 5^x$, *giving your answer in exact form.*

Step 1: Take the natural log of both sides:

$\ln(3^{2x+1}) = \ln(5^x)$

Step 2: Apply the power law:

$(2x+1)\ln 3 = x \ln 5$

Step 3: Expand and collect x terms:

$2x \ln 3 + \ln 3 = x \ln 5 \ln 3 = x \ln 5 - 2x \ln 3 = x(\ln 5 - 2\ln 3)$

Step 4: Solve for x:

$x = \frac{\ln 3}{\ln 5 - 2\ln 3} = \frac{\ln 3}{\ln 5 - \ln 9} = \frac{\ln 3}{\ln(5/9)}$

Q4.

Finding a specific term in a binomial expansion

Find the term containing x^3 *in the expansion of* $\left(2x - \dfrac{1}{x}\right)^9$.

Step 1: Write the general term with $a = 2x$ and $b = -\dfrac{1}{x}$:

$T_{r+1} = \binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x}\right)^r = \binom{9}{r}2^{9-r}(-1)^r \cdot x^{9-r} \cdot x^{-r}$

Step 2: Combine powers of x:

$T_{r+1} = \binom{9}{r}2^{9-r}(-1)^r \cdot x^{9-2r}$

Step 3: Set the power of x equal to 3:

$9 - 2r = 3 \implies r = 3$

Step 4: Substitute $r = 3$:

$T_4 = \binom{9}{3}2^6(-1)^3 \cdot x^3 = 84 \times 64 \times (-1) \cdot x^3 = -5376\, x^3$

Q5.

Binomial expansion with constant term

Find the constant term in the expansion of $\left(x^2 + \dfrac{3}{x}\right)^6$.

Step 1: General term with $a = x^2$ and $b = \dfrac{3}{x}$:

$T_{r+1} = \binom{6}{r}(x^2)^{6-r}\left(\frac{3}{x}\right)^r = \binom{6}{r}3^r \cdot x^{12-2r} \cdot x^{-r} = \binom{6}{r}3^r \cdot x^{12-3r}$

Step 2: For a constant term, set power of x to zero:

$12 - 3r = 0 \implies r = 4$

Step 3: Substitute $r = 4$:

$T_5 = \binom{6}{4}3^4 = 15 \times 81 = 1215$

Q6.

Prove by induction that $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ **for all** $n \in \mathbb{Z}^+$**.**

State $P(n)$**:** $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Base case $(n = 1)$**:**

$\text{LHS} = 1$. $\text{RHS} = \dfrac{1 \times 2}{2} = 1$.

$\text{LHS} = \text{RHS}$, so $P(1)$ is true. ✓

Inductive hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.,

$\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$

Inductive step: We need to show $P(k+1)$ is true, i.e., that:

$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$

Starting from the LHS of $P(k+1)$:

$\sum_{r=1}^{k+1} r = \underbrace{\displaystyle\sum_{r=1}^{k} r}_{\text{use hypothesis}} + (k+1) = \frac{k(k+1)}{2} + (k+1)$

Factor out $(k+1)$:

$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1) \cdot \frac{k+2}{2} = \frac{(k+1)(k+2)}{2}$

This is the RHS of $P(k+1)$. ✓

Conclusion: Since $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{Z}^+$. ∎

Q7.

Prove by induction that $4^n - 1$ **is divisible by 3 for all** $n \in \mathbb{Z}^+$**.**

State $P(n)$**:** $3 \mid (4^n - 1)$, i.e., $4^n - 1 = 3m$ for some integer m.

Base case ($n = 1$):

$4^1 - 1 = 3 = 3 \times 1$. This is divisible by 3. $P(1)$ is true. ✓

Inductive hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.,

$$4^k - 1 = 3m \quad \text{for some integer } m, \text{ so} \quad 4^k = 3m + 1$$

Inductive step: We need to show $P(k+1)$ is true, i.e., $3 \mid (4^{k+1} - 1)$.

$$4^{k+1} - 1 = 4 \cdot 4^k - 1$$

Substitute the inductive hypothesis ($4^k = 3m + 1$):

$$= 4(3m + 1) - 1 = 12m + 4 - 1 = 12m + 3 = 3(4m + 1)$$

Since $4m + 1$ is an integer, $4^{k+1} - 1$ is divisible by 3. $P(k+1)$ is true. ✓

Conclusion: Since $P(1)$ is true, and $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$, by the principle of mathematical induction $4^n - 1$ is divisible by 3 for all $n \in \mathbb{Z}^+$. ■

Q8.

Gaussian elimination — inconsistent system (no solution)

Solve:

$x + 2y - z = 4$ $2x - y + 3z = 1$ $3x + y + 2z = 11$

Step 1: Write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 1 \\ 3 & 1 & 2 & 11 \end{array}\right)$$

Step 2: Eliminate x from rows 2 and 3.

$R_2 \to R_2 - 2R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 3 & 1 & 2 & 11 \end{array}\right)$$

$R_3 \to R_3 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & -5 & 5 & -1 \end{array}\right)$$

Step 3: Eliminate y from row 3.

$R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -5 & 5 & -7 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

Step 4: The last row reads $0 = 6$, a **contradiction**. This system has **no solution** (the planes are inconsistent).

Q9.

Review Set 14A — Q1

On an Argand diagram, illustrate the complex numbers: (a) $3 - 2i$ *(b)* $-1 + 5i$

Q10.

Review Set 14A — Q3

Find $|z|$ *and* $\arg z$ *for: (a)* $5 + 2i$ *(b)* $-5 + 2i$ *(c)* $-5 - 2i$ *(d)* $-2 - 5i$

Worked Solution — Modulus and argument, four cases

All four share the same modulus: $|z| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$

Step a: $z = 5 + 2i \to \text{Q1} \to \arg = \arctan\left(\frac{2}{5}\right) \approx 0.381$ rad (positive real, positive imag)

Step b: $z = -5 + 2i \to \text{Q2} \to \arg = \pi - \arctan\left(\frac{2}{5}\right) \approx 2.761$ rad (negative real, positive imag)

Step c: $z = -5 - 2i \to \text{Q3} \to \arg = -\pi + \arctan\left(\frac{2}{5}\right) \approx -2.761$ rad (negative real, negative imag)

Step d: $z = -2 - 5i \to \text{Q3} \to \arg = -\pi + \arctan\left(\frac{5}{2}\right) \approx -1.951$ rad (note $|b/a| = 5/2$, not $2/5$)

Q11.

Review Set 14A — Q2

$z = 3 + 2i$*,* $w = -2 + i$*. Find: (a)* $z + w$ *(b)* $2z - w$ *(c)* z^* *(d)* $3w^* - z^*$

Worked Solution — Arithmetic with two complex numbers

Step a: $z + w = (3 - 2) + (2 + 1)i = 1 + 3i$ (add real parts; add imag parts)

Step b: $2z - w = (6 + 4i) - (-2 + i) = 8 + 3i$ (expand $2z = 6 + 4i$, then subtract)

Step c: $z^* = 3 - 2i$ (flip sign of imaginary part only)

Step d: $w^* = -2 - i \to 3w^* = -6 - 3i$ (conjugate flips $+i$ to $-i$)

$3w^* - z^* = (-6 - 3i) - (3 - 2i) = -9 - i$ (subtract term by term)

Q12.

Review Set 14A — Q4

Given $|z| = 5$, *find: (a)* $|5i|$ *(b)* $|-4z^*|$ *(c)* $|(3+i)z|$ *(d)* $|i/z|$ *(e)* $|3/z|$ *(f)* $|(3+6i)/z|$

Worked Solution — Modulus algebra (rules: $|zw| = |z||w|$**,** $|z/w| = |z|/|w|$**,** $|z^*| = |z|$**)**

Core rules: $|z \cdot w| = |z| \cdot |w|, |z/w| = |z|/|w|, |z^*| = |z|, |i| = 1$

Step a: $|5i| = 5 \cdot |i| = 5 \cdot 1 = 5$ ($|5| = 5, |i| = 1$)

Step b: $|-4z^*| = 4 \cdot |z^*| = 4 \cdot 5 = 20$ ($|-4| = 4, |z^*| = |z| = 5$)

Step c: $|(3+i)z| = |3+i| \cdot |z| = \sqrt{10} \cdot 5 = 5\sqrt{10}$ ($|3+i| = \sqrt{9+1} = \sqrt{10}$)

Step d: $|i/z| = 1/5$ ($|i| = 1, |z| = 5$)

Step e: $|3/z| = 3/5$

Step f: $|(3+6i)/z| = \sqrt{45}/5 = \frac{3\sqrt{5}}{5}$ ($|3+6i| = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$)

Q13.

Review Set 14A — Q13

$z = 4+i, w = 2-3i$. *Find: (a)* $2w^* - iz$ *(b)* $|w - z^*|$ *(c)* $|z^{100}|$ *(d)* $\arg(w-z)$

Worked Solution — Mixed operations including high power

$z = 4+i, w = 2-3i, z^* = 4-i, w^* = 2+3i$

Step a:

$iz = i(4+i) = 4i + i^2 = -1+4i$ (multiply out, $i^2 = -1$)

$2w^* = 2(2+3i) = 4+6i$

$2w^* - iz = (4+6i) - (-1+4i) = 5+2i$ (subtract)

Step b:

$w - z^* = (2-3i) - (4-i) = -2-2i$

$|w - z^*| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ (Pythagoras)

Step c:

$|z^{100}| = |z|^{100} = (\sqrt{17})^{100} = 17^{50}$ ($|z| = \sqrt{16+1} = \sqrt{17}$)

Step d:

$w - z = -2 - 4i \to$ Q3

$\arg(w - z) = -\pi + \arctan\left(\frac{4}{2}\right) = -\pi + \arctan(2) \approx -2.034$ rad (Q3 formula)

Q14.

Worked Conversion: $-1 + i\sqrt{3} \to$ Polar

Step 1: $a = -1, b = \sqrt{3} \to$ sketch shows Q2 ($a < 0, b > 0$) — identify quadrant

Step 2: $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ — modulus by Pythagoras

Step 3: Reference angle $= \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}$ — arctan of $|b/a|$

Step 4: Q2 formula: $\arg = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ — apply quadrant shift

Step 5: Answer: $z = 2 \cdot e^{i \cdot 2\pi/3} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ — write polar form

Q15.

Review Set 14A — Q11

Write $-1 + i\sqrt{3}$ in polar form. Hence find all values of n for which $(-1 + i\sqrt{3})^n$ is real.

Worked Solution — Polar form then condition for real power

Polar form (from conversion above): $z = 2 \cdot e^{i \cdot 2\pi/3}$

Step 1: $z^n = 2^n \cdot e^{i \cdot n \cdot 2\pi/3} = 2^n\left(\cos\frac{2\pi n}{3} + i\sin\frac{2\pi n}{3}\right)$ — de Moivre

Step 2: For z^n to be REAL: imaginary part $= 0$, so $\sin\left(\frac{2\pi n}{3}\right) = 0$ — condition for real

Step 3: $\sin(k\pi) = 0$ for any integer k, so $\frac{2\pi n}{3} = k\pi \to n = \frac{3k}{2}$ — solve for n

Step 4: For INTEGER n: $\frac{3k}{2}$ is integer only when k is even ($k = 2m$)

Step 5: $n = 3m$ for integer m. **Answer: n is any multiple of 3**

Q16.

Review Set 14A — Q15

Write $2 - 2\sqrt{3}\,i$ in polar form. Hence find all values of n for which $(2 - 2\sqrt{3}\,i)^n$ is purely imaginary.

Worked Solution — Polar form then condition for purely imaginary

Step 1: $r = \sqrt{4 + 12} = \sqrt{16} = 4$. Point $(2, -2\sqrt{3})$ is Q4.

Step 2: $\arg = -\arctan\left(\frac{2\sqrt{3}}{2}\right) = -\arctan(\sqrt{3}) = -\frac{\pi}{3}$ (Q4: negate arctan)

Step 3: Polar: $z = 4 \cdot e^{-i\pi/3} = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$

Step 4: $z^n = 4^n \cdot \text{cis}\left(-\frac{n\pi}{3}\right)$. For purely imaginary: $\text{Re}(z^n) = 0$ — condition

Step 5: $\cos\left(-\frac{n\pi}{3}\right) = 0 \to \frac{n\pi}{3} = \frac{\pi}{2} + k\pi \to n = \frac{3}{2} + 3k$ — solve

Step 6: Not an integer for any k. Check period: arguments cycle every 6 steps.

Step 7: $n = 1$: $4\,\text{cis}(-\pi/3)$, $n = 2$: $16\,\text{cis}(-2\pi/3)$, $n = 3$: $64\,\text{cis}(-\pi) = -64$ [real, not imag] — direct check

Step 8: No positive integer n gives a purely imaginary result for this z.

Q17.

Review Set 14A — Q17

$z = 4\,\text{cis}\,\theta$. Find the modulus and argument of: (a) z^3 (b) $1/z$ (c) iz^n

Worked Solution — Polar operations on $z = 4\,\text{cis}\,\theta$

Given: $|z| = 4$, $\arg(z) = \theta$. Apply polar rules:

Step a: z^3: $|z^3| = 4^3 = 64$, $\arg(z^3) = 3\theta$ (modulus cubed, arg tripled)

Step b: $1/z$: $|1/z| = 1/4$, $\arg(1/z) = -\theta$ (negate argument for reciprocal)

Step c: iz^n: $|i| = 1$, $\arg(i) = \pi/2$

$|iz^n| = 1 \cdot 4^n = 4^n$, $\arg(iz^n) = \frac{\pi}{2} + n\theta$ (add arguments)

Q18.

Review Set 14A — Q19/20

Use polar form to show $|1/z| = 1/|z|$ and $\arg(1/z) = -\arg z$. Write $1 + i$ in polar form.

Worked Solution — Polar proof and conversion

Step 1: Let $z = r \cdot e^{i\theta}$. Then $1/z = (1/r) \cdot e^{-i\theta}$ — reciprocal in polar

Step 2: $|1/z| = 1/r = 1/|z|$ ✓ — modulus of $(1/r)e^{-i\theta}$ is $1/r$

Step 3: $\arg(1/z) = -\theta = -\arg(z)$ ✓ — argument is $-\theta$

Step 4: $1 + i$: $r = \sqrt{2}$, Q1: $\arg = \arctan(1/1) = \pi/4$ — conversion

Step 5: $1 + i = \sqrt{2} \cdot e^{i\pi/4} = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$ — polar form

Q19.

Review Set 14A — Q6

If $(x+iy)^n = X + Yi$ *where* n *is a positive integer, show that* $X^2 + Y^2 = (x^2 + y^2)^n$.

Worked Solution — De Moivre modulus proof

Step 1: Write $z = x + iy$ in polar form: $z = r \cdot e^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ — convert to polar

Step 2: By de Moivre: $z^n = r^n \cdot e^{in\theta} = X + Yi$ — apply theorem

Step 3: Read off Cartesian parts: $X = r^n \cos(n\theta), Y = r^n \sin(n\theta)$ — real and imag parts

Step 4: $X^2 + Y^2 = r^{2n} \cos^2(n\theta) + r^{2n} \sin^2(n\theta) = r^{2n}(\cos^2(n\theta) + \sin^2(n\theta))$ — sum of squares

Step 5: $= r^{2n} \cdot 1 = r^{2n} = \left(\sqrt{x^2+y^2}\right)^{2n} = (x^2 + y^2)^n$ ✓ — substitute r, done

Q20.

Review Set 14A — Q10

$z = 4 \operatorname{cis} \theta$. *Find modulus and argument of: (a)* $(2z)^{-1}$ *(b)* $1 - z$

Worked Solution — Combined polar operations

Step a: $(2z)^{-1}$: $|2z| = 2 \cdot 4 = 8, \arg(2z) = \theta$ ($\times 2$ is real, adds 0 to arg)

$|(2z)^{-1}| = \frac{1}{8}, \arg((2z)^{-1}) = -\theta$ — negate argument for reciprocal

Step b: $z = 4\cos\theta + 4i\sin\theta, 1 - z = (1 - 4\cos\theta) - 4i\sin\theta$ — convert to Cartesian then subtract

$|1 - z| = \sqrt{(1 - 4\cos\theta)^2 + 16\sin^2\theta}$ — Pythagoras

$\arg(1 - z) = \arctan\left(\frac{-4\sin\theta}{1 - 4\cos\theta}\right)$ [use correct quadrant] — argument

Q21.

Oxford Key — Q14a

Express $\cos^5\theta$ *in terms of* $\cos(n\theta)$ *using the* $z + 1/z$ *technique*.

Worked Solution — Full expansion step by step

Step 1: Set $z = e^{i\theta}$. Then $(z + z^{-1})^5 = (2\cos\theta)^5 = 32\cos^5\theta$ — setup

Step 2: Expand $(z + z^{-1})^5$ by binomial theorem (coefficients $1, 5, 10, 10, 5, 1$):

$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

Step 3: Group symmetric pairs:

$= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

Step 4: Apply $z^n + z^{-n} = 2\cos(n\theta)$ to each group:

$= 2\cos(5\theta) + 5 \cdot 2\cos(3\theta) + 10 \cdot 2\cos(\theta) = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

Step 5: Equate: $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$ — set equal to right side

Step 6: Divide by 32:

$\cos^5\theta = \frac{1}{16}\left[\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right] \quad \checkmark$

Q22.

Oxford Key — Q14b

Hence evaluate $\displaystyle\int \cos^5\theta\, d\theta$.

Worked Solution — Integration using the multiple-angle form

$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta) \quad$ [from Q14a above]

$\int \cos^5\theta\, d\theta = \frac{1}{16}\int(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\, d\theta$

Step 1: $\displaystyle\int \cos 5\theta\, d\theta = \frac{\sin 5\theta}{5}$

Step 2: $\displaystyle\int 5\cos 3\theta\, d\theta = \frac{5\sin 3\theta}{3}$ (coefficient 5 stays)

Step 3: $\displaystyle\int 10\cos\theta\, d\theta = 10\sin\theta$

Step 4: Combine:

$\frac{1}{16}\left[\frac{\sin 5\theta}{5} + \frac{5\sin 3\theta}{3} + 10\sin\theta\right] + C$

Step 5: Simplify:

$\frac{\sin 5\theta}{80} + \frac{5\sin 3\theta}{48} + \frac{5\sin\theta}{8} + C$

(verified by differentiation ✓)

Q23.

Oxford Key — Trig Identity

Use de Moivre's theorem to prove $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ *and* $\sin(2\theta) = 2\sin\theta\cos\theta$.

Worked Solution — Double angle identities from de Moivre

Step 1: $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$ [de Moivre, $n = 2$] — apply theorem

Step 2: Expand left side:

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$

Step 3: Equate real parts: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ ✓

Step 4: Equate imaginary parts: $\sin(2\theta) = 2\sin\theta\cos\theta$ ✓

Q24.

Review Set 14A — Q9

Find the cube roots of $-64i$, giving answers in Cartesian form.

Worked Solution — Cube roots of $-64i$

Step 1: $-64i$ lies on the negative imaginary axis: $r = 64, \arg = -\frac{\pi}{2}$ — polar form

Step 2: Write: $-64i = 64 \cdot e^{i(-\pi/2)}$

Step 3: Modulus of each root: $64^{1/3} = 4$ — cube root of 64

Step 4: Arguments: $\theta_k = \frac{-\pi/2 + 2\pi k}{3}$ for $k = 0, 1, 2$ — apply formula

- $k = 0$: $\theta = -\frac{\pi}{6}$
- $k = 1$: $\theta = -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{2}$
- $k = 2$: $\theta = -\frac{\pi}{6} + \frac{4\pi}{3} = \frac{7\pi}{6} \to$ use $-\frac{5\pi}{6}$ (adjust to $(-\pi, \pi]$)

Step 5: Convert to Cartesian:

- $k = 0$: $4 \cdot \text{cis}\left(-\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = 2\sqrt{3} - 2i$
- $k = 1$: $4 \cdot \text{cis}\left(\frac{\pi}{2}\right) = 4(0 + i) = 4i$
- $k = 2$: $4 \cdot \text{cis}\left(-\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -2\sqrt{3} - 2i$

Step 6: Verify: $(2\sqrt{3} - 2i)^3 = -64i$ ✓ (check by expansion)

Q25.

Review Set 14A — Q16

Determine the cube roots of -27.

Worked Solution — Cube roots of -27

Step 1: -27 is on the negative real axis: $r = 27, \arg = \pi$

Step 2: Write: $-27 = 27 \cdot e^{i\pi}$

Step 3: Modulus of each root: $27^{1/3} = 3$

Step 4: Arguments: $\theta_k = \frac{\pi + 2\pi k}{3}$ for $k = 0, 1, 2$

- $k = 0$: $\theta = \frac{\pi}{3} \to 3 \cdot \text{cis}\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}\,i$
- $k = 1$: $\theta = \pi \to 3 \cdot \text{cis}(\pi) = 3(-1+0) = -3$
- $k = 2$: $\theta = \frac{5\pi}{3} \to 3 \cdot \text{cis}\left(\frac{5\pi}{3}\right) = 3\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}\,i$

Q26.

Review Set 14A — Q22

State the five fifth roots of unity and hence solve: (a) $(2z-1)^5 = 32$ *(b)* $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z = 0$ *(c)* $(z+1)^5 = (z-1)^5$

Worked Solution — Fifth roots of unity and applications

Fifth roots of unity: $z_k = e^{2\pi ik/5}$ for $k = 0, 1, 2, 3, 4$

$= 1,\ e^{2\pi i/5},\ e^{4\pi i/5},\ e^{6\pi i/5},\ e^{8\pi i/5}$

Their sum $= 0$ (Vieta's formulas on $z^5 - 1 = 0$) ✓

Step a: $(2z-1)^5 = 32 = 2^5 \cdot 1 \to (2z-1)^5 = 2^5$

$2z - 1 = 2 \cdot z_k \to z = \frac{1 + 2\cdot e^{2\pi ik/5}}{2}$ for $k = 0, 1, 2, 3, 4$ — 5 solutions

Step b: $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z = 0 \to z(z^4 + 5z^3 + 10z^2 + 10z + 5) = 0$ — factor out z

Remaining factor: note this is related to the $(z+1)^5$ expansion.

$z = 0$ is one solution; remaining 4 from $z^4 + 5z^3 + \cdots = 0$

Step c: $(z+1)^5 = (z-1)^5 \to \left(\frac{z+1}{z-1}\right)^5 = 1$ [divide both sides] — $z \neq 1$

Let $w = \frac{z+1}{z-1}$. Then $w^5 = 1$, so $w = e^{2\pi ik/5}$ for $k = 0, 1, 2, 3, 4$

$w = 1$ ($k = 0$) gives $\frac{z+1}{z-1} = 1 \to$ no solution. Use $k = 1, 2, 3, 4$.

For each k: $z = \frac{w+1}{w-1}$ where $w = e^{2\pi ik/5}$ — 4 solutions

Q27.

Oxford Key — Exam-Style

$(1+i)^{10}$ *— find its value. (Using de Moivre)*

Worked Solution — $(1+i)^{10}$ using de Moivre

Step 1: Write $1 + i$ in polar: $r = \sqrt{2}$, arg $= \frac{\pi}{4} \to 1 + i = \sqrt{2} \cdot e^{i\pi/4}$

Step 2: $(1+i)^{10} = (\sqrt{2})^{10} \cdot e^{i\cdot 10\pi/4}$ by de Moivre

Step 3: $(\sqrt{2})^{10} = 2^5 = 32$

Step 4: $\frac{10\pi}{4} = \frac{5\pi}{2} = 2\pi + \frac{\pi}{2} \to$ equivalent angle $= \frac{\pi}{2}$ — reduce to $(-\pi, \pi]$

Step 5: $= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 32(0+i) = 32i$ — **final answer**

Q28.

Review Set 14A — Q14

Find the Cartesian equation of the locus of $P(x,y)$ *where* $z = x + iy$ *and: (a)* $\arg(z-i) = \frac{\pi}{6}$ *(b)* $\left|\frac{z+2}{z-2}\right| = 2$

Worked Solution — Two loci derived from scratch

PART (a): $\arg(z-i) = \frac{\pi}{6}$

Step 1: $z - i = x + (y-1)i$ — substitute $z = x + iy$

Step 2: $\arg = \frac{\pi}{6}$ means $\tan\left(\frac{\pi}{6}\right) = \frac{\text{Im}}{\text{Re}} = \frac{y-1}{x}$ — tan of argument

Step 3: $\frac{1}{\sqrt{3}} = \frac{y-1}{x} \to x = \sqrt{3}(y-1) \to y = \frac{x}{\sqrt{3}} + 1$ — solve for y

Step 4: This is a **RAY** at angle $\frac{\pi}{6}$, starting from $(0,1)$, going into $x > 0$ — add constraint

PART (b): $\left|\frac{z+2}{z-2}\right| = 2$

Step 1: $|z+2| = 2|z-2| \to |z+2|^2 = 4|z-2|^2$ — square both sides

Step 2: $(x+2)^2 + y^2 = 4[(x-2)^2 + y^2]$ — expand moduli

Step 3: $x^2 + 4x + 4 + y^2 = 4x^2 - 16x + 16 + 4y^2$ — expand brackets

Step 4: $0 = 3x^2 - 20x + 12 + 3y^2$ — rearrange

Step 5: $x^2 - \frac{20}{3}x + 4 + y^2 = 0 \to \left(x - \frac{10}{3}\right)^2 + y^2 = \frac{100}{9} - \frac{36}{9} = \frac{64}{9}$ — complete the square

Step 6: Circle: centre $\left(\frac{10}{3}, 0\right)$, radius $= \frac{8}{3}$ — final answer

Q29.

Review Set 14A — Q18

Illustrate the region defined by: $\{z \mid 2 \le |z| \le 5 \text{ and } -\frac{\pi}{4} < \arg z \le \frac{\pi}{2}\}$. *Show all included boundary points.*

Worked Solution — Region defined by modulus AND argument constraints

Condition 1: $2 \le |z| \le 5 \to$ the region BETWEEN two circles

- $|z| = 2$: inner circle (solid boundary — included because $\ge$)
- $|z| = 5$: outer circle (solid boundary — included because $\le$)

Condition 2: $-\frac{\pi}{4} < \arg(z) \le \frac{\pi}{2} \to$ a sector (wedge)

- $\arg = -\frac{\pi}{4}$: lower boundary (DASHED — not included because $<$)
- $\arg = \frac{\pi}{2}$: upper boundary (SOLID — included because $\leq$)

Combined region: a sector of an annulus (ring)

Draw the two arcs and two radii, mark which boundaries are solid vs dashed.

Q30.

Review Set 14A — Q21

$P_1P_2P_3$ is an isosceles triangle where $P_1P_3 = \sqrt{3} \cdot P_1P_2$. O is origin. $OP_1 = z_1$, $OP_2 = z_2$, $OP_3 = z_3$. $\arg(z_3 - z_1) = \alpha$. (a) Show $\arg(z_3 - z_2) = \alpha - \frac{\pi}{2}$ (b) Find modulus and argument of $\frac{z_3 - z_1}{z_3 - z_2}$

Worked Solution — Complex numbers as position vectors in geometry

Step 1: $z_3 - z_1$ and $z_3 - z_2$ are the vectors $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_2P_3}$ respectively — interpret geometrically

Step 2: $P_1P_3 = \sqrt{3} \cdot P_1P_2$ means $|z_3 - z_1| = \sqrt{3}\,|z_3 - z_2|$ — given length condition

Step 3: $P_1P_2P_3$ isosceles: angle at $P_3 = 90^\circ$ (since $P_1P_3 = \sqrt{3} \cdot P_1P_2$ and isosceles) — geometry

Step 4: $\arg(z_3 - z_2) = \arg(z_3 - z_1) - \frac{\pi}{2} = \alpha - \frac{\pi}{2}$ — perpendicularity gives $-\pi/2$

Step 5: $\left|\frac{z_3 - z_1}{z_3 - z_2}\right| = \sqrt{3}$, $\arg = \alpha - \left(\alpha - \frac{\pi}{2}\right) = \frac{\pi}{2}$ — divide the complex numbers

So $\frac{z_3 - z_1}{z_3 - z_2} = \sqrt{3} \cdot e^{i\pi/2} = i\sqrt{3}$ — Cartesian form

Q31.

IB Problem — Q1a [2 marks]

Show that $z_1 z_2^* = r_1 r_2 e^{i(\alpha - \theta)}$ where z_2^* is the complex conjugate of z_2.

Worked Solution — Product with conjugate in polar form

Step 1: $z_1 = r_1 e^{i\alpha}$ and $z_2^* = r_2 e^{-i\theta}$ [conjugate negates argument] — write in polar

Step 2: $z_1 \cdot z_2^* = r_1 e^{i\alpha} \cdot r_2 e^{-i\theta}$ — multiply

Step 3: $= r_1 r_2 \cdot e^{i(\alpha - \theta)}$ ✓ — add exponents: $i\alpha + (-i\theta) = i(\alpha - \theta)$

Q32.

IB Problem — Q1b [2 marks]

Given $\mathrm{Re}(z_1 z_2^*) = 0$, show that $Z_1 O Z_2$ is a right-angled triangle.

Worked Solution — $\mathrm{Re} = 0$ means perpendicular

Step 1: From Q1a: $z_1 z_2^* = r_1 r_2 \cdot e^{i(\alpha-\theta)}$

Step 2: $\text{Re}(z_1 z_2^*) = r_1 r_2 \cdot \cos(\alpha - \theta)$ — real part of $re^{i\varphi}$ is $r \cos \varphi$

Step 3: $\text{Re} = 0$ and $r_1, r_2 > 0 \to \cos(\alpha - \theta) = 0 \to \alpha - \theta = \frac{\pi}{2}$ — solve for angle

Step 4: The angle at O between OZ_1 and OZ_2 is $\frac{\pi}{2} = 90^\circ$ — geometric conclusion

Step 5: Therefore $Z_1 O Z_2$ has a right angle at O ✓

Q33.

IB Problem — Q1c.i [2 marks]

For equilateral triangle $Z_1 O Z_2$: express z_1 in terms of z_2.

Worked Solution — Equilateral triangle condition

Step 1: Equilateral: all three sides equal, so $|z_1| = |z_2| \to r_1 = r_2$ — all sides equal

Step 2: Equilateral also means all angles $= 60^\circ$, so angle at $O = \frac{\pi}{3}$ — all angles 60 degrees

Step 3: $\alpha - \theta = \frac{\pi}{3} \to \alpha = \theta + \frac{\pi}{3}$ — angle between z_1 and z_2

Step 4: $z_1 = r_1 e^{i\alpha} = r_2 e^{i(\theta+\pi/3)} = r_2 e^{i\theta} \cdot e^{i\pi/3} = z_2 \cdot e^{i\pi/3}$ — factor

Step 5: $e^{i\pi/3} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ — exact value

Step 6: $z_1 = z_2\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ ✓ — final form

Q34.

IB Problem — Q1c.ii [4 marks]

Hence show that $z_1^2 + z_2^2 = z_1 z_2$.

Worked Solution — Algebraic identity for equilateral triangle

Step 1: From c.i: $z_1 = z_2 \cdot e^{i\pi/3}$, so let $w = z_1/z_2 = e^{i\pi/3}$ — divide

Step 2: $z_1^2 + z_2^2 = z_1 z_2 \longleftrightarrow$ divide both sides by z_2^2:

$(z_1/z_2)^2 + 1 = (z_1/z_2) \to w^2 - w + 1 = 0$ — in terms of w

Step 3: Verify: $w = e^{i\pi/3}, w^2 = e^{2i\pi/3}$

$w^2 - w + 1 = e^{2i\pi/3} - e^{i\pi/3} + 1$

$= \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + 1 = \left(-\frac{1}{2} - \frac{1}{2} + 1\right) + i\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$ ✓ — compute

Q35.

IB Problem — Q1d [5 marks]

z_1 and z_2 are roots of $z^2 + az + b = 0$ ($a, b \in \mathbb{R}$). Use result from (c)(ii) to show $a^2 - 3b = 0$.

Worked Solution — Vieta's formulas + equilateral condition

Step 1: Vieta's formulas for $z^2 + az + b = 0$:

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ — sum and product of roots

Step 2: From c.ii: $z_1^2 + z_2^2 = z_1 z_2$ — use proven result

Step 3: $(z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$ — expand $z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1 z_2$

Step 4: $(-a)^2 - 2b = b$ — substitute Vieta

Step 5: $a^2 - 2b = b \to a^2 = 3b \to a^2 - 3b = 0$ ✓ — rearrange

Q36.

IB Problem — Q1e [3 marks]

$z^2 + az + 12 = 0$, $a \in \mathbb{R}$. Given $0 < \alpha - \theta < \pi$, deduce only one equilateral triangle Z_1OZ_2 can be formed.

Worked Solution — Uniqueness of equilateral triangle

Step 1: $b = 12$ (constant term of quadratic). From d: $a^2 = 3b = 36$, so $a = \pm 6$

Step 2: Both values give a valid quadratic, but check which gives $0 < \alpha - \theta < \pi$

Step 3: $a = -6$: $z^2 - 6z + 12 = 0 \to z = \frac{6 \pm \sqrt{36-48}}{2} = 3 \pm i\sqrt{3}$ — quadratic formula

$z_1 = 3 + i\sqrt{3}$: $r = 2\sqrt{3}, \alpha = \frac{\pi}{6}$. $z_2 = 3 - i\sqrt{3}$: $r = 2\sqrt{3}, \theta = -\frac{\pi}{6}$

$\alpha - \theta = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$. Is $0 < \frac{\pi}{3} < \pi$? YES ✓

Step 4: $a = 6$: $z^2 + 6z + 12 = 0 \to z = \frac{-6 \pm i\sqrt{12}}{2} = -3 \pm i\sqrt{3}$

$z_1 = -3 + i\sqrt{3}$: $\alpha = \frac{5\pi}{6}$. $z_2 = -3 - i\sqrt{3}$: $\theta = -\frac{5\pi}{6}$ (or $\frac{7\pi}{6}$)

$\alpha - \theta = \frac{5\pi}{6} - \left(-\frac{5\pi}{6}\right) = \frac{5\pi}{3}$. Is $0 < \frac{5\pi}{3} < \pi$? NO — exceeds π — fails constraint

Step 5: Only $a = -6$ satisfies $0 < \alpha - \theta < \pi$. **Exactly one equilateral triangle.** ✓

Q37.

IB Problem — Q2a(i) [4 marks]

Determine the value of $1 + \omega + \omega^2$.

Q38.

IB Problem — Q2a(ii) [4 marks]

Determine the value of $1 + \omega^* + (\omega^*)^2$.

Worked Solution — Using $\omega^* = \omega^2$

Step 1: $\omega^* = \omega^2$ (conjugate of a non-real cube root of unity is the other one) — key fact

Step 2: $1 + \omega^* + (\omega^*)^2 = 1 + \omega^2 + (\omega^2)^2 = 1 + \omega^2 + \omega^4$ — substitute

Step 3: $\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$ — power cycle

Step 4: $= 1 + \omega^2 + \omega = 1 + \omega + \omega^2 = 0$ ✓ — apply main identity

Q39.

IB Problem — Q2b [4 marks]

Show that $(\omega - 3\omega^2)(\omega^2 - 3\omega) = 13$.

Worked Solution — Expand and simplify using $1 + \omega + \omega^2 = 0$

Step 1: Expand: $(\omega - 3\omega^2)(\omega^2 - 3\omega)$

$= \omega \cdot \omega^2 - 3\omega \cdot \omega - 3\omega^2 \cdot \omega^2 + 9\omega^2 \cdot \omega$ — FOIL

$= \omega^3 - 3\omega^2 - 3\omega^4 + 9\omega^3$ — collect terms

Step 2: Substitute $\omega^3 = 1$ and $\omega^4 = \omega$:

$= 1 - 3\omega^2 - 3\omega + 9 \cdot 1$

$= 10 - 3(\omega + \omega^2)$ — factor

Step 3: Substitute $\omega + \omega^2 = -1$:

$= 10 - 3(-1) = 10 + 3 = 13$ ✓ — apply identity

Q40.

IB Problem — Q2c [5 marks]

$p = 1 - 3i$, $q = x + (2x + 1)i$, $x \in \mathbb{R}$. *Find all values of* x *such that* $|p| = |q|$.

Worked Solution — Equal moduli equation

Step 1: $|p|^2 = 1^2 + (-3)^2 = 1 + 9 = 10$ — compute $|p|$

Step 2: $|q|^2 = x^2 + (2x + 1)^2 = x^2 + 4x^2 + 4x + 1 = 5x^2 + 4x + 1$ — expand $|q|$

Step 3: Set $|p|^2 = |q|^2$: $5x^2 + 4x + 1 = 10$ — equal moduli

Step 4: $5x^2 + 4x - 9 = 0$ — rearrange

Step 5: Discriminant: $16 + 4(5)(9) = 16 + 180 = 196 = 14^2$

Step 6: $x = \frac{-4 \pm 14}{10} \to x = \frac{10}{10} = 1$ or $x = \frac{-18}{10} = -\frac{9}{5}$ — quadratic formula

Step 7: Verify: $x = 1$: $|q|^2 = 5 + 4 + 1 = 10$ ✓. $x = -\frac{9}{5}$: $\frac{5 \cdot 81}{25} + \frac{4 \cdot (-9)}{5} + 1 = \frac{81}{5} - \frac{36}{5} + \frac{5}{5} = \frac{50}{5} = 10$ ✓

Q41.

IB Problem — Q2d [6 marks]

Solve the inequality $\mathrm{Re}(pq) + 8 < (\mathrm{Im}(pq))^2$.

Worked Solution — Compute pq then solve inequality

Step 1: $p = 1 - 3i, q = x + (2x+1)i$ — given

Step 2: $pq = (1 - 3i)(x + (2x+1)i)$ — multiply out

$= x + (2x+1)i - 3xi - 3i(2x+1)i$ — expand

$= x + (2x+1)i - 3xi + 3(2x+1)$ [since $-3i \cdot (2x+1)i = -3(2x+1)i^2 = +3(2x+1)$] — $i^2 = -1$

$= [x + 6x + 3] + [(2x+1) - 3x]i$ — collect real and imag

$= (7x+3) + (1-x)i$ — final form

Step 3: $\mathrm{Re}(pq) = 7x + 3, \mathrm{Im}(pq) = 1 - x$ — read off parts

Step 4: Inequality: $(7x+3) + 8 < (1-x)^2$ — substitute

Step 5: $7x + 11 < 1 - 2x + x^2$ — expand right side

Step 6: $0 < x^2 - 9x - 10 \to 0 < (x-10)(x+1)$ — rearrange and factor

Step 7: Parabola $(x-10)(x+1) > 0$ when $x < -1$ or $x > 10$ — sign analysis

Step 8: Solution: $x < -1$ **or** $x > 10$

Q42.

Question 1 [De Moivre's Theorem] [~7 marks]

(a) Express $z = 1 + i$ in modulus-argument (polar) form $r(\cos\theta + i\sin\theta)$.

(b) Use de Moivre's theorem to find $z^6 = (1+i)^6$. Give your answer in the form $a + bi$.

(c) Hence find the value of $(1+i)^6 + (1-i)^6$.

▶ Show Solution

Q43.

Question 2 [Roots of Unity] [~8 marks]

(a) Find all fifth roots of unity, expressing each in the form $e^{i\theta}$.

(b) Plot the five roots on an Argand diagram and describe the geometric figure they form.

(c) Show that the sum of all fifth roots of unity equals zero.

(d) Find the exact value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$.

▶ Show Solution

Q44.

Question 3 [Loci in the Complex Plane] [~6 marks]

A complex number z satisfies $|z - 3 - 4i| = |z + 1 - 2i|$.

(a) Describe the locus of z geometrically.

(b) Find the equation of this locus in the form $y = mx + c$, where $z = x + iy$.

(c) Determine whether the point $z = 2 + 5i$ lies on this locus.

▶ Show Solution

Q45.

Finding domain and range

Find the domain and range of $f(x) = \sqrt{4 - x^2}$.

Domain: Require $4 - x^2 \geq 0$, i.e. $x^2 \leq 4$, so $-2 \leq x \leq 2$.

$$\text{Domain: } [-2,\, 2]$$

Range: The expression $\sqrt{4 - x^2}$ is the upper half of a circle of radius 2. The maximum occurs at $x = 0$ (giving $\sqrt{4} = 2$) and the minimum is 0 at $x = \pm 2$.

$$\text{Range: } [0,\, 2]$$

Q46.

Finding a composite and its domain

Let $f(x) = \sqrt{x}$ (domain $x \geq 0$) and $g(x) = 3 - x^2$ (domain $\mathbb{R}$). Find $(f \circ g)(x)$ and its domain.

Step 1: $(f \circ g)(x) = f(g(x)) = \sqrt{3 - x^2}$

Step 2: Domain requires $3 - x^2 \geq 0$, so $x^2 \leq 3$, giving $-\sqrt{3} \leq x \leq \sqrt{3}$.

$$\text{dom}(f \circ g) = [-\sqrt{3},\, \sqrt{3}]$$

Q47.

Finding an inverse function

Find $f^{-1}(x)$ *for* $f(x) = \dfrac{2x+1}{x-3}$, $x \neq 3$.

Step 1: Let $y = \dfrac{2x+1}{x-3}$.

Step 2: Rearrange for x:

$y(x-3) = 2x+1$ $xy - 3y = 2x+1$ $xy - 2x = 3y+1$ $x(y-2) = 3y+1$ $x = \frac{3y+1}{y-2}$

Step 3: Swap $x \leftrightarrow y$:

$f^{-1}(x) = \frac{3x+1}{x-2}, \quad x \neq 2$

Verification: $f(f^{-1}(x)) = f\left(\dfrac{3x+1}{x-2}\right) = \dfrac{2 \cdot \frac{3x+1}{x-2} + 1}{\frac{3x+1}{x-2} - 3} = \dfrac{\frac{6x+2+x-2}{x-2}}{\frac{3x+1-3x+6}{x-2}} = \dfrac{7x}{7} = x$ ✓

Q48.

Restricting domain of $f(x) = x^2$

Find a restricted domain and the inverse of $f(x) = x^2$.

$f(x) = x^2$ is many-to-one over $\mathbb{R}$ (fails horizontal line test), so it has no inverse on $\mathbb{R}$.

Restriction: $\mathrm{dom}(f) = [0, \infty)$ makes f one-to-one.

Inverse: $y = x^2 \Rightarrow x = \sqrt{y}$ (taking the positive root because $x \geq 0$).

$f^{-1}(x) = \sqrt{x}, \quad x \geq 0$

Q49.

Describing a combined transformation

Describe the transformations that map $y = x^2$ *onto* $y = 3(x-2)^2 + 1$.

Reading from the form $y = a\,f(b(x-h)) + k$ with $f(x) = x^2$, $a = 3, b = 1, h = 2$, $k = 1$:

1. Vertical stretch by factor 3 (every y-value multiplied by 3).
2. Translation by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (2 units right, 1 unit up).

Note: vertical stretch before translation, or the translation remains correct regardless of order here since $b = 1$.

Q50.

Oblique asymptote by long division

Find the oblique asymptote of $f(x) = \dfrac{x^2+3x-2}{x+1}$.

Perform long division of x^2+3x-2 by $x+1$:

$x^2+3x-2 = (x+1)(x+2)-4$

Therefore:

$f(x) = x+2-\frac{4}{x+1}$

As $x \to \pm\infty$, $\dfrac{4}{x+1} \to 0$, so the oblique asymptote is $y = x+2$.

Q51.

Sketching $f(x) = \dfrac{x+1}{x-2}$

Step 1 — Vertical asymptote: $x-2=0 \Rightarrow x=2$. Numerator at $x=2$: $3 \neq 0$. Vertical asymptote: $x=2$.

Step 2 — Holes: No common factors. No holes.

Step 3 — Horizontal asymptote: Both numerator and denominator degree 1; leading coefficients both 1. Horizontal asymptote: $y=1$.

Step 4 — x-intercept: $x+1=0 \Rightarrow x=-1$. Point: $(-1,0)$.

Step 5 — y-intercept: $f(0) = \dfrac{1}{-2} = -\dfrac{1}{2}$. Point: $\left(0, -\dfrac{1}{2}\right)$.

Step 6 — Sign table:

Interval	Sign of $x+1$	Sign of $x-2$	Sign of f
$x<-1$	$-$	$-$	$+$
$-1<x<2$	$+$	$-$	$-$
$x>2$	$+$	$+$	$+$

Q52.

Solving $|2x-3|=5$

Case 1: $2x-3=5 \Rightarrow 2x=8 \Rightarrow x=4$

Case 2: $2x-3=-5 \Rightarrow 2x=-2 \Rightarrow x=-1$

Both solutions are valid. Answer: $x=4$ or $x=-1$.

Q53.

Solving $|x-1|=|2x+3|$

When both sides are moduli, square both sides (valid since both sides are ≥ 0):

$(x-1)^2 = (2x+3)^2$ $x^2 - 2x + 1 = 4x^2 + 12x + 9$ $3x^2 + 14x + 8 = 0$ $(3x+2)(x+4) = 0$ $x = -\frac{2}{3} \quad \text{or} \quad x = -4$

Verify: $|-\frac{2}{3} - 1| = \frac{5}{3}$ and $|2(-\frac{2}{3}) + 3| = \frac{5}{3}$ ✓. $|-4-1| = 5$ and $|2(-4)+3| = 5$ ✓.

Q54.

Solving $|3x - 6| \leq 9$

$-9 \leq 3x - 6 \leq 9$ $-3 \leq 3x \leq 15$ $-1 \leq x \leq 5$

Answer: $x \in [-1, 5]$.

Q55.

Solving $|x + 2| > 4$

Branch 1: $x + 2 > 4 \Rightarrow x > 2$

Branch 2: $x + 2 < -4 \Rightarrow x < -6$

Answer: $x \in (-\infty, -6) \cup (2, \infty)$.

Q56.

Polynomial long division

Divide $p(x) = x^3 - 6x^2 + 11x - 6$ *by* $(x-1)$.

Set up the division: $x^3 - 6x^2 + 11x - 6$ divided by $(x-1)$.

Step 1: Divide the leading term: $x^3 \div x = x^2$. Multiply: $x^2(x-1) = x^3 - x^2$. Subtract:

$$(x^3 - 6x^2 + 11x - 6) - (x^3 - x^2) = -5x^2 + 11x - 6$$

Step 2: Divide: $-5x^2 \div x = -5x$. Multiply: $-5x(x-1) = -5x^2 + 5x$. Subtract:

$$(-5x^2 + 11x - 6) - (-5x^2 + 5x) = 6x - 6$$

Step 3: Divide: $6x \div x = 6$. Multiply: $6(x-1) = 6x - 6$. Subtract:

$$(6x - 6) - (6x - 6) = 0$$

Result:

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6) + 0$$

The remainder is 0, confirming that $(x-1)$ is a factor. The quotient $x^2 - 5x + 6 = (x-2)(x-3)$, so $p(x) = (x-1)(x-2)(x-3)$.

Q57.

Finding all factors using the Factor Theorem

Factorise $p(x) = x^3 - 3x^2 - 4x + 12$ *completely.*

Step 1 — Test rational root candidates. By the rational root theorem, candidates are $\pm1, \pm2, \pm3, \pm6$.

$p(1) = 1 - 3 - 4 + 12 = 6 \neq 0$ $p(-1) = -1 - 3 + 4 + 12 = 12 \neq 0$ $p(2) = 8 - 12 - 8 + 12 = 0\checkmark$

So $(x - 2)$ is a factor.

Step 2 — Divide out the known factor. Divide $p(x)$ by $(x - 2)$:

$$x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 - x - 6)$$

Verify: $(x - 2)(x^2 - x - 6)$. Expand: $x^3 - x^2 - 6x - 2x^2 + 2x + 12 = x^3 - 3x^2 - 4x + 12$. Correct.

Step 3 — Factorise the quadratic.

$$x^2 - x - 6 = (x - 3)(x + 2)$$

Result:

$$p(x) = (x - 2)(x - 3)(x + 2)$$

Q58.

Applying the Remainder Theorem

Find the remainder when $p(x) = 2x^3 - x^2 + 3x - 5$ *is divided by* $(x + 2)$.

The divisor is $(x + 2) = (x - (-2))$, so $a = -2$. Evaluate:

$$p(-2) = 2(-2)^3 - (-2)^2 + 3(-2) - 5 = 2(-8) - 4 + (-6) - 5 = -16 - 4 - 6 - 5 = -31$$

The remainder is -31. No long division required.

Q59.

Solving a cubic equation completely

Solve $x^3 - 2x^2 - 5x + 6 = 0$.

Let $p(x) = x^3 - 2x^2 - 5x + 6$. Test rational root candidates $\pm1, \pm2, \pm3, \pm6$:

$$p(1) = 1 - 2 - 5 + 6 = 0\checkmark$$

So $(x - 1)$ is a factor. Divide $p(x)$ by $(x - 1)$:

$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$

Factorise the quadratic:

$x^2 - x - 6 = (x-3)(x+2)$

Therefore:

$p(x) = (x-1)(x-3)(x+2)$

Solutions: $x = 1, x = 3, x = -2.$

Check: $p(3) = 27 - 18 - 15 + 6 = 0$ and $p(-2) = -8 - 8 + 10 + 6 = 0.$ Confirmed.

Q60.

Distance in 3D

Find the distance between $P(1,-2,4)$ *and* $Q(5,0,1)$*, and the midpoint of* PQ*.*

$d = \sqrt{(5-1)^2 + (0-(-2))^2 + (1-4)^2} = \sqrt{16+4+9} = \sqrt{29}$

$M = \left(\frac{1+5}{2},\ \frac{-2+0}{2},\ \frac{4+1}{2}\right) = \left(3,\ -1,\ \frac{5}{2}\right)$

Q61.

Composite Solid — Cone on Cylinder

A solid consists of a cylinder of radius 4 cm and height 6 cm, with a cone of the same base radius and height 3 cm placed on top. Find the total volume and the total surface area.

Volume:

$V = \pi(4)^2(6) + \frac{1}{3}\pi(4)^2(3) = 96\pi + 16\pi = 112\pi \approx 352\ \text{cm}^3$

Surface area: The solid has a circular base (bottom of cylinder), the curved lateral surface of the cylinder, and the curved lateral surface of the cone. The circle where cylinder meets cone is internal — it does not contribute.

Slant height of cone: $l = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$ cm

$SA = \pi(4)^2 + 2\pi(4)(6) + \pi(4)(5) = 16\pi + 48\pi + 20\pi = 84\pi \approx 264\ \text{cm}^2$

Q62.

Sine Rule — Finding a Side

In triangle ABC*, angle* $A = 35^\circ$*, angle* $B = 72^\circ$*, and side* $a = 8.4$ *cm. Find side* b*.*

First find $C = 180^\circ - 35^\circ - 72^\circ = 73^\circ$.

$\frac{b}{\sin B} = \frac{a}{\sin A}$

$b = \frac{8.4 \times \sin 72^\circ}{\sin 35^\circ} = \frac{8.4 \times 0.9511}{0.5736} \approx 13.9 \text{ cm}$

Q63.

Cosine Rule — Finding an Angle

A triangle has sides $a = 7, b = 5, c = 9$. *Find angle* C.

$\cos C = \frac{7^2+5^2-9^2}{2(7)(5)} = \frac{49+25-81}{70} = \frac{-7}{70} = -0.1$

$C = \arccos(-0.1) \approx 95.7^\circ$

Since $\cos C < 0$, the angle C is obtuse — this is consistent with c being the longest side.

Q64.

Bearings Problem

A ship travels 12 km on a bearing of 050°, *then 9 km on a bearing of* 140°. *Find the distance and bearing back to the start.*

The two legs form a triangle. The angle between the directions 050° and 140° is $140^\circ - 50^\circ = 90^\circ$ (the turn at the intermediate point is 90°, making the paths perpendicular).

$d = \sqrt{12^2 + 9^2} = \sqrt{144 + 81} = \sqrt{225} = 15 \text{ km}$

The angle at the start satisfies $\tan\theta = \frac{9}{12} = 0.75$, so $\theta = 36.87^\circ \approx 36.9^\circ$.

Bearing from start to final position: $050^\circ + 36.9^\circ = 086.9^\circ \approx 087^\circ$.

Return bearing: $087^\circ + 180^\circ = 267^\circ$.

Q65.

Arc Length and Sector Area

A sector of a circle has radius 8 cm and central angle $\dfrac{5\pi}{6}$ *radians. Find the arc length, the sector area, and the area of the corresponding minor segment.*

Arc length: $l = r\theta = 8 \times \dfrac{5\pi}{6} = \dfrac{40\pi}{6} = \dfrac{20\pi}{3} \approx 20.9 \text{ cm}$

Sector area: $A = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(64)\left(\dfrac{5\pi}{6}\right) = \dfrac{320\pi}{12} = \dfrac{80\pi}{3} \approx 83.8 \text{ cm}^2$

Segment area: $A_{\text{seg}} = \dfrac{1}{2}r^2(\theta - \sin\theta) = 32\left(\dfrac{5\pi}{6} - \sin\dfrac{5\pi}{6}\right) = 32\left(\dfrac{5\pi}{6} - \dfrac{1}{2}\right) \approx 32(2.618 - 0.5) \approx 67.8 \text{ cm}^2$

Q66.

Compound Angle Identity — Exact Value

Show that $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Write $75^\circ = 45^\circ + 30^\circ$:

$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$ ✓

Q67.

Identity Proof

Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

Starting from the left-hand side:

$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$ ✓

Q68.

Graph Transformations

Describe the transformations of $y = -3\cos\left(2x - \dfrac{\pi}{3}\right) + 1$ *and state the amplitude, period, phase shift, and range*.

Rewrite as $y = -3\cos\left(2\left(x - \dfrac{\pi}{6}\right)\right) + 1$.

- $a = -3$: amplitude 3, reflected in x-axis
- $b = 2$: period $= \dfrac{2\pi}{2} = \pi$
- $c = \dfrac{\pi}{6}$: shifted right by $\dfrac{\pi}{6}$
- $d = 1$: shifted up by 1; midline $y = 1$

Range: $[1-3,\ 1+3] = [-2, 4]$

Q69.

Solving a Trig Equation on an Interval

Solve $2\cos^2 x - \cos x - 1 = 0$ *for* $0 \le x \le 2\pi$.

Step 1: Factorise — treat $\cos x$ as the variable.

$2\cos^2 x - \cos x - 1 = (2\cos x + 1)(\cos x - 1) = 0$

Step 2: Set each factor to zero.

$\cos x = -\dfrac{1}{2}$ or $\cos x = 1$

Step 3: Solve each case on $[0, 2\pi]$.

For $\cos x = 1$: $x = 0$ (and $x = 2\pi$, but check interval is $0 \leq x \leq 2\pi$, so $x = 0$ and $x = 2\pi$ are both valid endpoints).

For $\cos x = -\dfrac{1}{2}$: reference angle $= \dfrac{\pi}{3}$; cosine is negative in Q2 and Q3.

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \qquad \text{or} \qquad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

Solutions: $x \in \left\{0,\ \dfrac{2\pi}{3},\ \dfrac{4\pi}{3},\ 2\pi\right\}$

Q70.

Trig Equation Requiring an Identity

Solve $\sin 2x = \sin x$ *for* $0 \leq x < 2\pi$.

Step 1: Apply the double angle identity: $\sin 2x = 2\sin x \cos x$.

$2\sin x \cos x = \sin x$

Step 2: Rearrange (do not divide by $\sin x$ — you would lose solutions!).

$2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

Step 3: Solve each factor.

$\sin x = 0 \implies x = 0,\ \pi$

$\cos x = \dfrac{1}{2} \implies x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Solutions: $x \in \left\{0,\ \dfrac{\pi}{3},\ \pi,\ \dfrac{5\pi}{3}\right\}$

Q71.

Reciprocal Trig Identity Proof

Prove that $\dfrac{\csc\theta}{\cot\theta + \tan\theta} = \cos\theta$.

LHS:

$$\frac{\csc\theta}{\cot\theta+\tan\theta} = \frac{\dfrac{1}{\sin\theta}}{\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}}$$

Simplify the denominator using a common denominator of $\sin\theta\cos\theta$:

$$= \frac{\frac{1}{\sin\theta}}{\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}} = \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta\cos\theta}} = \tfrac{1}{\sin\theta} \times \sin\theta\cos\theta = \cos\theta \qquad \checkmark$$

Q72.

Dot Product — Angle Between Vectors

Find the angle between $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$.

$\mathbf{u} \cdot \mathbf{v} = (2)(1) + (-1)(4) + (3)(-2) = 2 - 4 - 6 = -8$

$|\mathbf{u}| = \sqrt{4+1+9} = \sqrt{14} \qquad |\mathbf{v}| = \sqrt{1+16+4} = \sqrt{21}$

$\cos\theta = \frac{-8}{\sqrt{14}\cdot\sqrt{21}} = \frac{-8}{\sqrt{294}}$

$\theta = \arccos\left(\frac{-8}{\sqrt{294}}\right) \approx 117.8^\circ$

Q73.

Cross Product — Normal Vector and Area

Given $A(1,0,2)$, $B(3,1,0)$, $C(0,2,1)$, *find a vector normal to the plane* ABC *and the area of triangle* ABC.

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (1)(-1) - (-2)(2) \\ (-2)(-1) - (2)(-1) \\ (2)(2) - (1)(-1) \end{pmatrix} = \begin{pmatrix} -1+4 \\ 2+2 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Normal vector: $\mathbf{n} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

Area of triangle: $A = \frac{1}{2}\left|\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right| = \frac{1}{2}\sqrt{9+16+25} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2}$

Q74.

Line Intersection in 3D

Line L_1*:* $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ *and line* L_2*:* $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. *Determine whether the lines intersect, are parallel, or are skew.*

Step 1: Check if direction vectors are parallel. $(1,-1,2)$ and $(-1,1,0)$ are not scalar multiples (since $\frac{1}{-1} = -1$ but $\frac{2}{0}$ is undefined), so the lines are not parallel.

Step 2: Set corresponding components equal.

$1 + s = 3 - t \quad \Rightarrow \quad s + t = 2 \qquad (i)$ $2 - s = t \quad \Rightarrow \quad s + t = 2 \qquad (ii)$ $0 + 2s = 4 \quad \Rightarrow \quad s = 2 \qquad (iii)$

From (iii): $s = 2$. From (i): $t = 0$.

Step 3: Verify in the z-component of L_2: $z = 4 + 0(0) = 4$. For L_1: $z = 0 + 2(2) = 4$. Consistent.

The lines intersect at the point: L_1 at $s = 2$: $(1 + 2, 2 - 2, 0 + 4) = (3, 0, 4)$. Intersection point: $(3, 0, 4)$.

Q75.

Equation of a Plane Through Three Points

Find the Cartesian equation of the plane through $P(1, 0, 2)$*,* $Q(3, 1, 0)$*,* $R(0, 2, 1)$.

Step 1: Find two vectors in the plane.

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \overrightarrow{PR} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Step 2: Cross product gives the normal.

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} (1)(-1) - (-2)(2) \\ (-2)(-1) - (2)(-1) \\ (2)(2) - (1)(-1) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Step 3: Use point $P(1, 0, 2)$ to find d.

$$d = 3(1) + 4(0) + 5(2) = 3 + 0 + 10 = 13$$

Equation of the plane: $3x + 4y + 5z = 13$

Verification with $Q(3, 1, 0)$*:* $9 + 4 + 0 = 13$ ✓*. With* $R(0, 2, 1)$*:* $0 + 8 + 5 = 13$ ✓*.*

Q76.

Line-Plane Intersection

Find the point where L*:* $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ *meets the plane* Π*:* $x + 2y - z = 5$

.

Parametric equations of L**:** $x = 1 + 2t,\ y = 2 - t,\ z = -1 + 3t$

Substitute into plane equation:

$$(1 + 2t) + 2(2 - t) - (-1 + 3t) = 5$$

$1 + 2t + 4 - 2t + 1 - 3t = 5$

$6 - 3t = 5 \implies t = \frac{1}{3}$

Intersection point: $x = 1 + \frac{2}{3} = \frac{5}{3}, y = 2 - \frac{1}{3} = \frac{5}{3}, z = -1 + 1 = 0.$

Point of intersection: $\left(\frac{5}{3}, \frac{5}{3}, 0\right)$

Q77.

Intersection of Three Planes

Find the intersection of the planes $\Pi_1 : x + y + z = 6, \Pi_2 : 2x - y + z = 3, \Pi_3 : x + 2y - z = 4.$

Set up the system:

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + 2y - z &= 4 \end{aligned}$$

Eliminate x: Using $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - R_1$:

$$\begin{aligned} x + y + z &= 6 \\ -3y - z &= -9 \\ y - 2z &= -2 \end{aligned}$$

From row 2: $3y + z = 9$ so $z = 9 - 3y$.

Substitute into row 3: $y - 2(9 - 3y) = -2 \implies y - 18 + 6y = -2 \implies 7y = 16 \implies y = \dfrac{16}{7}$

Then: $z = 9 - 3\left(\dfrac{16}{7}\right) = 9 - \dfrac{48}{7} = \dfrac{15}{7}, x = 6 - y - z = 6 - \dfrac{16}{7} - \dfrac{15}{7} = 6 - \dfrac{31}{7} = \dfrac{11}{7}$

Intersection point: $\left(\dfrac{11}{7}, \dfrac{16}{7}, \dfrac{15}{7}\right)$

Q78.

Question 1 [Non-Right-Angle Trig and Sector] [~8 marks]

Triangle ABC has $AB = 10$ cm, $AC = 7$ cm, and angle $BAC = 1.2$ radians.

(a) Find the area of triangle ABC.

(b) Find the length BC.

(c) A sector of a circle has the same area as triangle ABC and a radius of 5 cm. Find the central angle of the sector in radians.

▶ Show Solution

Q79.

Question 2 [Trig Identity Proof and Equation] [~9 marks]

(a) Prove that $\cos 2\theta + 2\sin^2\theta = 1$.

(b) Hence, or otherwise, solve $\cos 4x + 2\sin^2 2x - 1 = 0$ for $0 \le x \le \pi$.

▶ Show Solution

Q80.

Question 3 [Compound Angle — HL] [~9 marks]

(a) Given that $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{5}{13}$, where α and β are both acute angles, find the exact value of $\cos(\alpha+\beta)$.

(b) Show that $\cos(\alpha+\beta) = \cos(\alpha-\beta) - 2\sin\alpha\sin\beta$.

▶ Show Solution

Q81.

Question 4 [Vectors — Lines and Planes] [~15 marks] HL

The plane Π_1 passes through the points $A(2,0,1)$, $B(1,3,-1)$, and $C(0,1,2)$.

(a) Find the equation of plane Π_1 in the form $ax + by + cz = d$.

A second plane Π_2 has equation $2x - y + z = 4$.

(b) Find the angle between Π_1 and Π_2.

(c) Find the vector equation of the line L of intersection of Π_1 and Π_2.

A point P lies on L with $x = 0$.

(d) Find the coordinates of P.

▶ Show Solution

Q82.

Question 5 [3D Trig Application] [~9 marks]

A vertical tower AB stands on horizontal ground at A. From a point C on the ground, the angle of elevation of the top B is 42°. The bearing of A from C is 325°, and $CA = 80$ m. A second point D on the ground is 50 m due East of C.

(a) Find the height AB of the tower.

(b) Find the angle of elevation of B from D.

(c) Find the bearing of A from D.

▶ Show Solution

Q83.

Combined Probability — Addition Rule

In a class of 30 students, 18 study French, 12 study Spanish, and 6 study both. A student is chosen at random. Find the probability they study French or Spanish.

Define events: Let F = studies French, S = studies Spanish.

$P(F) = \frac{18}{30}, \quad P(S) = \frac{12}{30}, \quad P(F \cap S) = \frac{6}{30}$

Apply the addition rule:

$P(F \cup S) = \frac{18}{30} + \frac{12}{30} - \frac{6}{30} = \frac{24}{30} = \frac{4}{5}$

Q84.

Venn Diagram — Finding Unknown Probabilities

Given $P(A) = 0.5$, $P(B) = 0.4$, and $P(A \cup B) = 0.7$, find $P(A \cap B)$.

Rearrange the addition rule:

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.4 - 0.7 = 0.2$

Hence find $P(A\ \text{only})$ — the probability of A but not B.

$P(A \cap B') = P(A) - P(A \cap B) = 0.5 - 0.2 = 0.3$

Q85.

Conditional Probability — Two-Way Table

A survey records whether 100 students exercise regularly and whether they sleep more than 7 hours per night.

	Sleep ≥ 7 h	**Sleep < 7 h**	**Total**
Exercises regularly	42	18	60
Does not exercise	23	17	40
Total	65	35	100

(a) Find the probability a randomly selected student exercises regularly given they sleep at least 7 hours.

(b) Find the probability a student sleeps fewer than 7 hours given they do not exercise regularly.

Part (a): Let E = exercises, G = sleeps ≥ 7 h.

$P(E \mid G) = \frac{P(E \cap G)}{P(G)} = \frac{42/100}{65/100} = \frac{42}{65} \approx 0.646$

Part (b): Let L = sleeps < 7 h, E' = does not exercise.

$P(L \mid E') = \frac{17/100}{40/100} = \frac{17}{40} = 0.425$

Q86.

Testing Independence

$P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.2$. *Are A and B independent?*

Check: $P(A) \times P(B) = 0.4 \times 0.5 = 0.20$

Since $P(A \cap B) = 0.20 = P(A) \cdot P(B)$, the events are **independent**.

Also verify: $P(A \mid B) = \dfrac{0.2}{0.5} = 0.4 = P(A)$. Confirmed.

Q87.

Tree Diagram — Two-Stage Problem

A box contains 4 red and 6 blue balls. Two balls are drawn without replacement. Find the probability that both balls are the same colour.

Stage 1 branch probabilities:

- $P(\text{Red}_1) = \frac{4}{10}$, $P(\text{Blue}_1) = \frac{6}{10}$

Stage 2 branch probabilities (conditional on Stage 1):

- After Red: $P(\text{Red}_2 \mid \text{Red}_1) = \frac{3}{9}$, $P(\text{Blue}_2 \mid \text{Red}_1) = \frac{6}{9}$
- After Blue: $P(\text{Red}_2 \mid \text{Blue}_1) = \frac{4}{9}$, $P(\text{Blue}_2 \mid \text{Blue}_1) = \frac{5}{9}$

Path probabilities (same colour):

$P(\text{RR}) = \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90}$

$P(\text{BB}) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90}$

$P(\text{same colour}) = \frac{12}{90} + \frac{30}{90} = \frac{42}{90} = \frac{7}{15}$

Q88.

Bayes' Theorem — Medical Test

A disease affects 1% of a population. A diagnostic test has a 95% sensitivity (true positive rate) and a 2% false positive rate. A randomly selected person tests positive. Find the probability they actually have the disease.

Define events:

- D = has the disease, D' = does not have the disease
- T^+ = tests positive

Given: $P(D) = 0.01, P(D') = 0.99, P(T^+ \mid D) = 0.95, P(T^+ \mid D') = 0.02$

Total probability of testing positive:

$$P(T^+) = P(T^+ \mid D) \cdot P(D) + P(T^+ \mid D') \cdot P(D')$$

$$= (0.95)(0.01) + (0.02)(0.99) = 0.0095 + 0.0198 = 0.0293$$

Apply Bayes:

$$P(D \mid T^+) = \frac{P(T^+|D) \cdot P(D)}{P(T^+)} = \frac{0.0095}{0.0293} \approx 0.324$$

Despite a positive test, there is only a 32.4% chance the person actually has the disease. This counter-intuitive result arises because the disease is rare — most positive tests come from the large pool of healthy people with false positives.

Q89.

Bayes' Theorem — Factory Quality Control

Machine A produces 60% of a factory's output; machine B produces the remaining 40%. Machine A has a 3% defect rate; machine B has a 5% defect rate. A randomly selected item is found to be defective. What is the probability it came from machine A?

Setup: $P(A) = 0.6, P(B) = 0.4, P(D \mid A) = 0.03, P(D \mid B) = 0.05$

$$P(D) = (0.03)(0.6) + (0.05)(0.4) = 0.018 + 0.020 = 0.038$$

$$P(A \mid D) = \frac{(0.03)(0.6)}{0.038} = \frac{0.018}{0.038} \approx 0.474$$

There is about a 47.4% chance the defective item came from machine A.

Q90.

Expected Value and Variance

A biased die has the following distribution:

x	1	2	3	4
$P(X = x)$	0.1	0.3	0.4	0.2

(a) Find $E(X)$. *(b) Find* $\text{Var}(X)$. *(c) Find* $E(3X - 2)$ *and* $\text{Var}(3X - 2)$.

Part (a):

$$E(X) = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2) = 0.1 + 0.6 + 1.2 + 0.8 = 2.7$$

Part (b): First compute $E(X^2)$:

$$E(X^2) = 1^2(0.1) + 2^2(0.3) + 3^2(0.4) + 4^2(0.2) = 0.1 + 1.2 + 3.6 + 3.2 = 8.1$$

$\mathrm{Var}(X) = 8.1 - (2.7)^2 = 8.1 - 7.29 = 0.81$

Part (c):

$E(3X - 2) = 3(2.7) - 2 = 8.1 - 2 = 6.1$

$\mathrm{Var}(3X - 2) = 3^2 \cdot \mathrm{Var}(X) = 9 \times 0.81 = 7.29$

Q91.

Binomial — Exact Probability

A fair coin is tossed 8 times. Find the probability of getting exactly 5 heads.

State the distribution: Let X = number of heads. Since trials are independent with $p = 0.5$ and $n = 8$: $X \sim B(8, 0.5)$.

$P(X = 5) = \binom{8}{5}(0.5)^5(0.5)^3 = 56 \times (0.5)^8 = \frac{56}{256} = \frac{7}{32} \approx 0.219$

Q92.

Binomial — Cumulative and Complement

In a production line, 15% of items are defective. A batch of 20 items is selected. Find: (a) $P(X \leq 3)$, (b) $P(X \geq 4)$, (c) $P(2 \leq X \leq 5)$.

State: $X \sim B(20, 0.15)$

Part (a): Use GDC cumulative binomial: $P(X \leq 3) \approx 0.6477$

Part (b): Complement rule:

$P(X \geq 4) = 1 - P(X \leq 3) \approx 1 - 0.6477 = 0.3523$

Part (c):

$P(2 \leq X \leq 5) = P(X \leq 5) - P(X \leq 1) \approx 0.9327 - 0.1756 = 0.7571$

Q93.

Binomial — Finding Parameters from Mean and Variance

A random variable $X \sim B(n, p)$ has mean 6 and variance 4.2. Find n and p.

Set up equations:

$np = 6 \qquad np(1 - p) = 4.2$

Divide the second by the first:

$1 - p = \frac{4.2}{6} = 0.7 \implies p = 0.3$

Substitute back:

$n(0.3) = 6 \implies n = 20$

Q94.

Poisson — Direct Calculation

Calls arrive at a helpdesk at an average rate of 3 per hour. Find the probability that: (a) exactly 2 calls arrive in one hour, (b) fewer than 4 calls arrive in one hour.

State: $X \sim \text{Po}(3)$

Part (a):

$P(X = 2) = \frac{e^{-3} \cdot 3^2}{2!} = \frac{e^{-3} \cdot 9}{2} \approx \frac{9}{2} \times 0.04979 \approx 0.224$

Part (b):

$P(X < 4) = P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= e^{-3}\left(1 + 3 + \frac{9}{2} + \frac{27}{6}\right) = e^{-3} \cdot 13 \approx 0.04979 \times 13 \approx 0.647$

Q95.

Poisson — Changing the Interval

Emails arrive at an average rate of 6 per hour. Find the probability that: (a) no emails arrive in 20 minutes, (b) more than 2 emails arrive in 30 minutes.

Key step: Rescale the rate to match the interval.

- In 20 minutes: $m = 6 \times \frac{20}{60} = 2$, so $X \sim \text{Po}(2)$
- In 30 minutes: $m = 6 \times \frac{30}{60} = 3$, so $Y \sim \text{Po}(3)$

Part (a):

$P(X = 0) = \frac{e^{-2} \cdot 2^0}{0!} = e^{-2} \approx 0.135$

Part (b):

$P(Y > 2) = 1 - P(Y \leq 2) = 1 - e^{-3}\left(1 + 3 + \frac{9}{2}\right) = 1 - e^{-3}(8.5) \approx 1 - 0.423 = 0.577$

Q96.

Poisson Approximation to Binomial HL

A rare genetic mutation occurs in 1 in 500 births. In a sample of 400 births, find the approximate probability that at most 2 have the mutation.

Check conditions: $n = 400$ (large), $p = 0.002$ (small). Approximate with $\text{Po}(m)$ where $m = np = 400 \times 0.002 = 0.8$.

$P(X \leq 2) = e^{-0.8}\left(1 + 0.8 + \frac{0.64}{2}\right) = e^{-0.8}(2.12) \approx 0.4493 \times 2.12 \approx 0.953$

Q97.

Normal Distribution — Finding Probabilities

Heights of adults are normally distributed with mean 170 cm and standard deviation 8 cm. Find: (a) $P(X > 180)$*, (b)* $P(160 < X < 185)$*.*

State: $X \sim N(170, 64)$

Part (a): $Z = \dfrac{180 - 170}{8} = 1.25$

Using GDC or standard normal table: $P(Z > 1.25) = 1 - P(Z \leq 1.25) \approx 1 - 0.8944 = 0.1056$

Part (b): Lower bound: $Z_1 = \dfrac{160 - 170}{8} = -1.25$; Upper bound: $Z_2 = \dfrac{185 - 170}{8} = 1.875$

$P(-1.25 < Z < 1.875) = P(Z < 1.875) - P(Z < -1.25) \approx 0.9696 - 0.1056 = 0.8640$

Q98.

Normal Distribution — Symmetry Shortcut

$X \sim N(50, 25)$*. Find* $P(45 < X < 55)$ *without a table.*

The interval $[45, 55]$ is symmetric about $\mu = 50$, each end $\frac{50-45}{5} = 1$ standard deviation away.

By the 68-95-99.7 rule: $P(\mu - \sigma < X < \mu + \sigma) \approx 0.683$.

For an exact answer: Z bounds are -1 and 1, so using GDC: $P(-1 < Z < 1) = 0.6827$.

Q99.

Paper 2 Style: Binomial vs Normal Approximation [8 marks]

A manufacturer produces electronic components, of which 5% are defective. A quality inspector tests a random sample of 200 components.

(a) Let X be the number of defective components found. Write down the exact distribution of X. [1]

(b) Verify that X can be approximated by a normal distribution and state the parameters of this normal distribution. [2]

(c) Using the normal approximation with continuity correction, find the probability that the inspector finds:

- (i) exactly 12 defective components [2]

- (ii) fewer than 8 defective components [3]

▶ Show Solution

Q100.

Inverse Normal — Finding a Threshold

Test scores are distributed as $X \sim N(65, 100)$*. The top 10% of students receive a distinction. Find the minimum score needed for a distinction.*

We need x such that $P(X > x) = 0.10$, i.e., $P(X \leq x) = 0.90$.

Using GDC: `invNorm(0.90, 65, 10)` $= 77.8$

A student needs at least **77.8** (round up to 78) to receive a distinction.

Q101.

Inverse Normal — Symmetric Interval

$X \sim N(\mu, \sigma^2)$ *with* $\mu = 40$ *and* $\sigma = 5$*. Find the value* k *such that* $P(\mu - k < X < \mu + k) = 0.90$.

By symmetry, $P(X < \mu - k) = 0.05$.

Using GDC: `invNorm(0.05, 40, 5)` ≈ 31.78, so $k = 40 - 31.78 = 8.22$.

Check: $P(40 - 8.22 < X < 40 + 8.22) = P(31.78 < X < 48.22) = 0.90$. Confirmed.

Q102.

Finding μ or σ from a Normal Probability

$X \sim N(\mu, 16)$*. It is given that* $P(X > 20) = 0.2$*. Find* μ*.*

Standardise: $P\left(Z > \dfrac{20 - \mu}{4}\right) = 0.2$

Find the z-score: $P(Z > z) = 0.2 \Rightarrow P(Z \leq z) = 0.8 \Rightarrow z \approx 0.8416$

Solve:

$\frac{20-\mu}{4} = 0.8416 \implies 20 - \mu = 3.366 \implies \mu \approx 16.6$

Q103.

Paper 2 Style: Type I and Type II Errors [6 marks]

A pharmaceutical company develops a new drug claimed to lower blood pressure. A clinical trial will test $H_0 : \mu = 140$ (no effect) vs. $H_1 : \mu < 140$ (drug lowers blood pressure), where μ is the mean systolic blood pressure (in mmHg) after treatment.

(a) Describe what a Type I error would mean in the context of this trial. [2]

(b) Describe what a Type II error would mean in the context of this trial. [2]

(c) The significance level is set at $\alpha = 0.05$. The medical regulator is concerned about approving ineffective drugs. Should they increase or decrease α? Explain the trade-off. [2]

▶ Show Solution

Q104.

Chi-Squared Goodness-of-Fit

A die is rolled 120 times. The observed frequencies are:

Face	1	2	3	4	5	6
Observed	17	22	18	25	19	19

Test at the 5% significance level whether the die is fair.

Step 1 — Hypotheses:

- H_0: The die is fair (each face has probability $\frac{1}{6}$)
- H_1: The die is not fair

Step 2 — Significance level: $\alpha = 0.05$

Step 3 — Expected frequencies: If fair, $E = \frac{120}{6} = 20$ for each face.

Step 4 — Test statistic:

$$\chi^2 = \frac{(17-20)^2}{20} + \frac{(22-20)^2}{20} + \frac{(18-20)^2}{20} + \frac{(25-20)^2}{20} + \frac{(19-20)^2}{20} + \frac{(19-20)^2}{20}$$

$$= \frac{9+4+4+25+1+1}{20} = \frac{44}{20} = 2.2$$

Step 5 — Degrees of freedom: $\nu = 6 - 1 = 5$

Step 6 — Critical value: $\chi^2_{5,0.05} = 11.07$

Step 7 — Decision: $\chi^2_{\text{calc}} = 2.2 < 11.07$, so we **fail to reject** H_0.

Conclusion: At the 5% significance level, there is insufficient evidence to conclude the die is unfair.

Q105.

Paper 2 Style: Chi-Squared Goodness-of-Fit Test [7 marks]

A biologist is studying the distribution of flower colors in a population of wildflowers. Mendelian genetics predicts the ratio Red : Pink : White should be 1:2:1. The biologist counts 240 flowers and observes:

Color	Red	Pink	White
Observed	52	136	52

(a) Calculate the expected frequencies under the genetic model. [2]

(b) State appropriate null and alternative hypotheses. [1]

(c) Calculate the χ^2 test statistic and state the degrees of freedom. [3]

(d) The critical value at the 5% significance level is 5.991. State your conclusion. [1]

▶ Show Solution

Q106.

Chi-Squared Test for Independence

Use the table from Section 1.2 (exercise vs. sleep). Test at 5% significance whether exercise and sleep are independent.

	Sleep $\geq$ 7 h	Sleep $<$ 7 h	Total
Exercises	42	18	60
Does not exercise	23	17	40
Total	65	35	100

Hypotheses: H_0: Exercise and sleep hours are independent. H_1: They are associated.

Expected frequencies: $E_{11} = \frac{60\times65}{100} = 39$, $E_{12} = \frac{60\times35}{100} = 21$, $E_{21} = \frac{40\times65}{100} = 26$, $E_{22} = \frac{40\times35}{100} = 14$

Test statistic:

$$\chi^2 = \frac{(42-39)^2}{39} + \frac{(18-21)^2}{21} + \frac{(23-26)^2}{26} + \frac{(17-14)^2}{14}$$

$$= \frac{9}{39} + \frac{9}{21} + \frac{9}{26} + \frac{9}{14} \approx 0.231 + 0.429 + 0.346 + 0.643 = 1.649$$

Degrees of freedom: $\nu = (2-1)(2-1) = 1$

Critical value: $\chi^2_{1,0.05} = 3.841$

Decision: $1.649 < 3.841$, fail to reject H_0.

Conclusion: At the 5% level, there is insufficient evidence of an association between exercise habits and sleep duration.

Q107.

One-Sample t-Test

A manufacturer claims their light bulbs last a mean of 1000 hours. A random sample of 15 bulbs gives a mean of 985 hours with a standard deviation of 30 hours. Test at the 5% level whether the mean lifetime is less than claimed.

Hypotheses: $H_0 : \mu = 1000$; $H_1 : \mu < 1000$ (one-tailed)

Significance level: $\alpha = 0.05$

Test statistic:

$$t = \frac{985-1000}{30/\sqrt{15}} = \frac{-15}{7.746} \approx -1.936$$

Degrees of freedom: $\nu = 15 - 1 = 14$

Critical value (one-tailed): $t_{14,0.05} = -1.761$

Decision: $t = -1.936 < -1.761$, so we **reject** H_0.

Conclusion: At the 5% significance level, there is sufficient evidence to conclude the mean bulb lifetime is less than 1000 hours.

Q108.

Paper 2 Style: Hypothesis Testing with t-distribution [8 marks]

A nutritionist claims that a new diet reduces cholesterol levels. A study measures cholesterol (in mmol/L) for 10 participants before and after 8 weeks on the diet.

Participant	1	2	3	4	5	6	7	8	9	10
Before	6.2	5.8	6.5	5.9	6.8	6.1	5.7	6.4	6.0	6.3
After	5.8	5.5	6.0	5.6	6.2	5.9	5.4	6.1	5.7	6.0

(a) State suitable null and alternative hypotheses to test whether the diet reduces cholesterol. [2]

(b) Calculate the differences for each participant (After – Before) and find the mean difference $\bar{d}$ and standard deviation s_d. [3]

(c) Perform a paired t-test at the 5% significance level and state your conclusion. [3]

▶ Show Solution

Q109.

Paper 2 Style: Two-Sample t-Test HL **[7 marks]**

An ecologist compares plant heights (in cm) between two different soil treatments. The data are:

Treatment A (organic): 24, 28, 22, 30, 26, 25, 29

Treatment B (synthetic): 32, 30, 35, 28, 34, 31

(a) State suitable hypotheses to test whether the mean height differs between treatments. [1]

(b) Calculate the sample means and sample standard deviations for both treatments. [2]

(c) The pooled variance is $s_p^2 = 11.27$. Perform a two-sample t-test at the 5% significance level. State the test statistic, degrees of freedom, and your conclusion. [4]

▶ Show Solution

Q110.

Paired t-Test HL

Eight students take a memory test before and after a training programme. Their scores are:

Student	1	2	3	4	5	6	7	8
Before	62	71	58	80	65	73	55	70
After	68	76	60	84	67	78	61	74

Test at the 5% level whether the training improves scores.

Step 1 — Compute differences d_i $=$ After $-$ Before:

| d_i | 6 | 5 | 2 | 4 | 2 | 5 | 6 | 4 |

Step 2 — Statistics of d:

$$\bar{d} = \frac{6+5+2+4+2+5+6+4}{8} = \frac{34}{8} = 4.25$$

$$s_d = \sqrt{\frac{\sum(d_i-\bar{d})^2}{n-1}} \approx 1.5811$$

Step 3 — Hypotheses: $H_0 : \mu_d = 0$; $H_1 : \mu_d > 0$ (one-tailed)

Step 4 — Test statistic:

$$t = \frac{4.25}{1.5811/\sqrt{8}} = \frac{4.25}{0.5590} \approx 7.60$$

Step 5 — Critical value: $t_{7,0.05} = 1.895$

Step 6 — Decision: $t = 7.60 \gg 1.895$, reject H_0.

Conclusion: At the 5% level, there is very strong evidence that the training programme significantly improves scores.

Q111.

Computing PMCC

Five students' hours of study (x) and exam marks (y) are:

Student	x	y
A	2	50
B	4	65
C	6	72
D	8	80
E	10	90

Calculate r.

Step 1 — Means: $\bar{x} = \frac{2+4+6+8+10}{5} = 6, \bar{y} = \frac{50+65+72+80+90}{5} = 71.4$

Step 2 — Sums of squares:

	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$
A	-4	-21.4	85.6	16	457.96
B	-2	-6.4	12.8	4	40.96
C	0	0.6	0	0	0.36
D	2	8.6	17.2	4	73.96
E	4	18.6	74.4	16	345.96
Sum			**190**	**40**	**919.2**

$$r = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}} = \frac{190}{\sqrt{40 \times 919.2}} = \frac{190}{\sqrt{36768}} = \frac{190}{191.75} \approx 0.991$$

Very strong positive linear correlation.

Q112.

Spearman's Rank — Step-by-Step

Six athletes are assessed by two judges. Their scores are:

Athlete	A	B	C	D	E	F
Judge 1 score	85	70	92	78	65	88
Judge 2 score	80	74	90	75	68	85

Calculate Spearman's rank correlation coefficient.

Step 1 — Rank each judge's scores (rank 1 = highest):

Athlete	Judge 1 score	Rank 1	Judge 2 score	Rank 2
A	85	3	80	3
B	70	5	74	4
C	92	1	90	1
D	78	4	75	5
E	65	6	68	6
F	88	2	85	2

Step 2 — Compute d_i and d_i^2:

Athlete	Rank 1	Rank 2	d_i	d_i^2
A	3	3	0	0
B	5	4	1	1
C	1	1	0	0
D	4	5	-1	1
E	6	6	0	0
F	2	2	0	0
Sum				**2**

Step 3 — Apply the formula ($n = 6, \sum d_i^2 = 2$):

$r_s = 1 - \frac{6\times2}{6(36-1)} = 1 - \frac{12}{210} = 1 - 0.0571 \approx 0.943$

Conclusion: $r_s \approx 0.943$ indicates very strong positive agreement between the two judges' rankings.

Q113.

Finding and Using the Regression Line

Using the data from the PMCC example above, find the regression line $y = ax + b$ and estimate the exam mark for a student who studies for 7 hours.

Step 1 — Calculate a:

$a = \frac{S_{xy}}{S_{xx}} = \frac{190}{40} = 4.75$

Step 2 — Calculate b:

$b = \bar{y} - a\bar{x} = 71.4 - 4.75 \times 6 = 71.4 - 28.5 = 42.9$

Regression line: $y = 4.75x + 42.9$

Prediction at $x = 7$:

$y = 4.75(7) + 42.9 = 33.25 + 42.9 = 76.15 \approx 76$

Q114.

Coefficient of Determination r^2

For the study data, $r \approx 0.991$. Interpret r^2.

$r^2 \approx (0.991)^2 \approx 0.982$

Interpretation: Approximately 98.2% of the variation in exam marks is explained by the linear relationship with hours of study. This suggests the linear model is an excellent fit for these data.

Q115.

Regression with GDC — Full Workflow

The following data shows temperature (x, in °C) and ice cream sales per day (y, in units):

x	15	18	22	25	28	30
y	80	110	145	170	200	220

(a) Find r and the regression line. (b) Estimate sales when temperature is 20°C. (c) Comment on reliability.

Part (a) — Using GDC (Casio):

1. Enter x values in List 1, y values in List 2
2. `STAT → CALC → 2VAR` to obtain: $\bar{x} = 23, \bar{y} = 154.17, r \approx 0.999$
3. `STAT → REG → ax+b`: $a \approx 9.77, b \approx -70.5$

Regression line: $y = 9.77x - 70.5$

Part (b): $y = 9.77(20) - 70.5 = 195.4 - 70.5 = 124.9 \approx 125$ units

Part (c): $r \approx 0.999$, which indicates a very strong positive linear correlation. Since $x = 20$ lies within the data range $[15, 30]$, this is interpolation and the prediction is reliable. The model explains $r^2 \approx 99.8\%$ of the variation in sales.

Q116.

Question 1 [Hypothesis Testing] [~8 marks]

A factory claims that the mean mass of its cereal boxes is 500 g. A quality inspector takes a random sample of 12 boxes and records the following masses (in grams):

$498,\ 502,\ 495,\ 501,\ 497,\ 503,\ 496,\ 499,\ 504,\ 498,\ 500,\ 497$

(a) State appropriate null and alternative hypotheses for a two-tailed test.

(b) The sample mean is $\bar{x} = 499.17$ g and the sample standard deviation is $s = 2.89$ g. Calculate the t-statistic for this test.

(c) The critical values for a two-tailed t-test at the 5% significance level with 11 degrees of freedom are ± 2.201. State the conclusion of the test, justifying your answer.

(d) State one assumption required for this test to be valid.

▶ Show Solution

Q117.

Question 2 [Bayes' Theorem] [~7 marks]

A medical screening test for a disease has the following properties:

- The probability of a positive result given the patient has the disease (sensitivity) is 0.95.
- The probability of a negative result given the patient does not have the disease (specificity) is 0.90.
- The prevalence of the disease in the population is 0.02.

(a) Construct a tree diagram or define the events and their probabilities.

(b) Find the probability that a randomly selected person tests positive.

(c) Find the probability that a person who tests positive actually has the disease.

(d) Comment on the practical implications of your answer to part (c).

▶ Show Solution

Q118.

Question 3 [Regression and Correlation] [~6 marks]

A researcher collects data on the number of hours spent studying (x) and exam score (y) for 8 students. The regression line of y on x is found to be:

$\hat{y} = 3.2x + 42.5$

The Pearson correlation coefficient is $r = 0.87$ and $\bar{x} = 6.5$.

(a) Interpret the value of the gradient (3.2) in context.

(b) Calculate the predicted exam score for a student who studies for 6.5 hours.

(c) Explain why it would be unreliable to use this model to predict the exam score for a student who studies for 20 hours.

(d) The coefficient of determination is r^2. Calculate r^2 and interpret it in context.

▶ Show Solution

Q119.

Limits by factoring

Evaluate $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

Step 1: Direct substitution gives $\frac{9-9}{3-3} = \frac{0}{0}$ — indeterminate form.

Step 2: Factor the numerator: $x^2 - 9 = (x-3)(x+3)$.

Step 3: Cancel: $\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$.

Step 4: Substitute: $\lim_{x \to 3}(x+3) = 6$

$\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$

Q120.

L'Hôpital's Rule

Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

Step 1: Direct substitution: $\frac{\sin 0}{0} = \frac{0}{0}$ — L'Hôpital applies.

Step 2: Differentiate numerator and denominator separately:

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$

This fundamental limit appears inside proofs of derivative formulas, so knowing the result $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ by heart saves time.

Q121.

Repeated L'Hôpital

Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Step 1: $\frac{e^0-1-0}{0} = \frac{0}{0}$ — apply L'Hôpital.

Step 2: $\lim\limits_{x \to 0} \dfrac{e^x - 1}{2x}$ — still $\frac{0}{0}$, apply again.

Step 3: $\lim\limits_{x \to 0} \dfrac{e^x}{2} = \dfrac{1}{2}$

Q122.

Derivative of $f(x) = x^2$ from first principles

Step 1: Write the difference quotient:

$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h}$

Step 2: Expand: $= \dfrac{x^2 + 2xh + h^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = 2x + h$

Step 3: Take the limit: $f'(x) = \lim\limits_{h \to 0}(2x + h) = 2x$

Q123.

Product Rule

Differentiate $y = x^3 e^{2x}$.

Let $u = x^3, v = e^{2x}$, so $u' = 3x^2, v' = 2e^{2x}$.

$\frac{dy}{dx} = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x} = e^{2x}(3x^2 + 2x^3) = x^2 e^{2x}(3 + 2x)$

Always factorise the final answer — IB mark schemes expect a simplified form.

Q124.

Quotient Rule

Differentiate $y = \dfrac{\sin x}{x^2+1}$.

$u = \sin x, v = x^2 + 1, u' = \cos x, v' = 2x.$

$\frac{dy}{dx} = \frac{\cos x(x^2+1) - \sin x \cdot 2x}{(x^2+1)^2} = \frac{(x^2+1)\cos x - 2x\sin x}{(x^2+1)^2}$

Q125.

Chain Rule

Differentiate $y = \sin(x^3 + 2x)$.

Let $u = x^3 + 2x$, so $y = \sin u$.

$\frac{dy}{dx} = \cos u \cdot \frac{du}{dx} = \cos(x^3 + 2x) \cdot (3x^2 + 2)$

Q126.

Nested Chain Rule

Differentiate $y = e^{\sin(2x)}$.

Layer 1 (outermost): $\frac{d}{du}(e^u) = e^u$ where $u = \sin(2x)$

Layer 2: $\frac{d}{dv}(\sin v) = \cos v$ where $v = 2x$

Layer 3: $\frac{d}{dx}(2x) = 2$

$\frac{dy}{dx} = e^{\sin(2x)} \cdot \cos(2x) \cdot 2 = 2\cos(2x)\, e^{\sin(2x)}$

Q127.

Implicit Differentiation

Find $\dfrac{dy}{dx}$ *for* $x^2 + y^2 = 25$.

Step 1: Differentiate both sides w.r.t. x:

$2x + 2y\frac{dy}{dx} = 0$

Step 2: Solve for $\frac{dy}{dx}$:

$\frac{dy}{dx} = -\frac{x}{y}$

This is the gradient at any point (x, y) on the circle. At $(3, 4)$: $\frac{dy}{dx} = -\frac{3}{4}$.

Q128.

Implicit Differentiation — Product Term

Find $\dfrac{dy}{dx}$ *for* $x^2y + y^3 = 5$.

Differentiate each term:

- $\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$ (product rule)
- $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$
- $\frac{d}{dx}(5) = 0$

Collect $\frac{dy}{dx}$ terms:

$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(x^2 + 3y^2) = -2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2+3y^2}$

Q129.

Inverse Trig with Chain Rule

Differentiate $y = \arctan(e^{2x})$.

Outer function: $\arctan(u)$ where $u = e^{2x}$ (inner function).

Apply the derivative formula for $\arctan$:

$\frac{dy}{dx} = \frac{1}{1+(e^{2x})^2} \cdot \frac{d}{dx}[e^{2x}]$

Differentiate the inner function using chain rule on the exponential:

$\frac{d}{dx}[e^{2x}] = e^{2x} \cdot 2 = 2e^{2x}$

Combine:

$\frac{dy}{dx} = \frac{2e^{2x}}{1+e^{4x}}$

This pattern appears frequently in Paper 2 when combining transcendental functions. A common error is forgetting to square the entire inner function in the denominator — it's $1 + (e^{2x})^2 = 1 + e^{4x}$, not $1 + e^{2x}$.

Q130.

Tangent and Normal

Find the equations of the tangent and normal to $y = x^3 - 2x$ *at the point where* $x = 2$.

Step 1: $y(2) = 8 - 4 = 4$ so the point is $(2, 4)$.

Step 2: $y' = 3x^2 - 2$, so $m_T = 3(4) - 2 = 10$.

Step 3: Tangent: $y - 4 = 10(x - 2) \Rightarrow y = 10x - 16$

Step 4: Normal: $m_N = -\frac{1}{10}$, so $y - 4 = -\frac{1}{10}(x - 2) \Rightarrow y = -\frac{x}{10} + \frac{21}{5}$

Q131.

Curve Sketching

Sketch $f(x) = \dfrac{x^2}{x^2 - 4}$*, identifying all key features.*

Domain: $x \neq \pm 2$

y-intercept: $f(0) = 0$

x-intercept: $x^2 = 0 \Rightarrow x = 0$

Vertical asymptotes: $x = 2$ and $x = -2$

Horizontal asymptote: As $x \to \pm\infty$, $f(x) \to 1$, so $y = 1$

Derivative: Using the quotient rule:

$f'(x) = \frac{2x(x^2-4) - x^2 \cdot 2x}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2}$

Stationary point at $x = 0$: $f(0) = 0$, $f''(0) > 0$ (local minimum at $(0, 0)$)

Behaviour: $f(x) > 1$ for $|x| > 2$; $f(x) \leq 0$ for $|x| < 2$.

Q132.

Optimisation — Closed Box

A closed rectangular box has a square base of side x cm and height h cm. Its surface area is 600 cm^2. Find the dimensions that maximise the volume.

Objective function: $V = x^2 h$

Constraint: $2x^2 + 4xh = 600 \Rightarrow h = \dfrac{600 - 2x^2}{4x} = \dfrac{300 - x^2}{2x}$

Single-variable form:

$V(x) = x^2 \cdot \frac{300-x^2}{2x} = \frac{x(300-x^2)}{2} = 150x - \frac{x^3}{2}$

Differentiate: $V'(x) = 150 - \frac{3x^2}{2}$

Set to zero: $150 = \frac{3x^2}{2} \Rightarrow x^2 = 100 \Rightarrow x = 10$ (positive dimension)

Confirm maximum: $V''(x) = -3x < 0$ for $x > 0$ — concave down, so this is a maximum.

Dimensions: $x = 10$ cm, $h = \frac{300-100}{20} = 10$ cm. The optimal box is a cube.

$V_{\max} = 1000 \text{ cm}^3$

Q133.

Related Rates — Expanding Circle

The radius of a circle is increasing at 3 cm s^{-1}*. Find the rate of increase of the area when the radius is* 5 *cm.*

Let r = radius, $A = \pi r^2$.

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

When $r = 5$ and $\frac{dr}{dt} = 3$:

$\frac{dA}{dt} = 2\pi(5)(3) = 30\pi \approx 94.2 \text{ cm}^2\text{s}^{-1}$

Q134.

Related Rates — Conical Water Tank (8 marks)

A water tank has the shape of an inverted circular cone with base radius 6 *m and height* 8 *m. Water is being pumped into the tank at a constant rate of* $2 \text{ m}^3\text{min}^{-1}$.

(a) [2 marks] *Express the radius* r *of the water surface in terms of the depth* h *of the water.*

(b) [3 marks] *Find an expression for the volume* V *of water in the tank in terms of* h *only.*

(c) [3 marks] *Find the rate at which the water level is rising when the depth is* 4 *m.*

Solution:

(a) By similar triangles, the ratio of radius to height is constant throughout the cone.

For the full tank: $\dfrac{6}{8} = \dfrac{3}{4}$

For the water at depth h:

$\frac{r}{h} = \frac{3}{4}$

Therefore:

$r = \frac{3h}{4}$

(b) The volume of a cone is given by:

$V = \frac{1}{3}\pi r^2 h$

Substituting $r = \dfrac{3h}{4}$:

$V = \frac{1}{3}\pi \left(\frac{3h}{4}\right)^2 h$

$V = \frac{1}{3}\pi \cdot \frac{9h^2}{16} \cdot h$

$V = \frac{3\pi h^3}{16}\ \mathrm{m}^3$

(c) Differentiate $V = \dfrac{3\pi h^3}{16}$ with respect to time t:

$\frac{dV}{dt} = \frac{3\pi}{16} \cdot 3h^2 \cdot \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{9\pi h^2}{16} \cdot \frac{dh}{dt}$

We are given that $\dfrac{dV}{dt} = 2\ \mathrm{m}^3\mathrm{min}^{-1}$ and we want to find $\dfrac{dh}{dt}$ when $h = 4$ m.

Substitute known values:

$2 = \frac{9\pi(4)^2}{16} \cdot \frac{dh}{dt}$

$2 = \frac{9\pi \cdot 16}{16} \cdot \frac{dh}{dt}$

$2 = 9\pi \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{2}{9\pi} \approx 0.0707\ \mathrm{m\ min}^{-1}$

The water level is rising at approximately $0.071\ \mathrm{m\ min}^{-1}$ (or $7.1\ \mathrm{cm\ min}^{-1}$) when the depth is 4 m.

Q135.

Kinematics

A particle has displacement $s(t) = t^3 - 6t^2 + 9t$ *metres,* $t \geq 0$ *seconds. Find: (a) when it is at rest, (b) its acceleration when* $t = 3$*, (c) the total distance in* $0 \leq t \leq 4$.

Part (a): $v = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0 \Rightarrow t = 1$ or $t = 3$

Part (b): $a = v' = 6t - 12$. At $t = 3$: $a = 18 - 12 = 6\ \mathrm{m\ s}^{-2}$

Part (c): Position values: $s(0) = 0, s(1) = 4, s(3) = 0, s(4) = 4$

Distances: $|4-0| + |0-4| + |4-0| = 4 + 4 + 4 = 12$ m

Q136.

Substitution — Indefinite

Find $\displaystyle\int 2x\cos(x^2)\,dx$.

Let $u = x^2$, so $du = 2x\,dx$.

$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C$

Q137.

Substitution — Definite

Evaluate $\int_0^1 \frac{x}{(x^2+1)^3}\,dx.$

Let $u = x^2+1, du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du.$

Change limits: $x = 0 \Rightarrow u = 1; x = 1 \Rightarrow u = 2.$

$\int_1^2 \frac{1}{u^3} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_1^2 u^{-3}\,du = \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^2 = -\frac{1}{4}\left[\frac{1}{4}-1\right] = -\frac{1}{4}\cdot\left(-\frac{3}{4}\right) = \frac{3}{16}$

Q138.

Integration by Parts — Standard

Find $\int xe^x\,dx.$

$u = x, dv = e^x\,dx \Rightarrow du = dx, v = e^x$

$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C$

Q139.

Integration by Parts — Twice (IBP^2)

Find $\int x^2e^x\,dx.$

First IBP: $u = x^2, dv = e^x\,dx \Rightarrow du = 2x\,dx, v = e^x$

$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx$

Second IBP: From above, $\int xe^x\,dx = e^x(x-1) + C$

$= x^2e^x - 2e^x(x-1) + C = e^x(x^2-2x+2) + C$

Q140.

Integration by Parts — Cyclic

Find $\int e^x \cos x\,dx.$

First IBP: $u = e^x, dv = \cos x\,dx \Rightarrow v = \sin x$

$I = e^x \sin x - \int e^x \sin x\,dx$

Second IBP: $u = e^x, dv = \sin x\,dx \Rightarrow v = -\cos x$

$I = e^x \sin x - \left(-e^x\cos x + \int e^x \cos x\,dx\right) = e^x \sin x + e^x \cos x - I$

Solve for I: $2I = e^x(\sin x + \cos x) \Rightarrow I = \dfrac{e^x(\sin x + \cos x)}{2} + C$

Q141.

Partial Fractions Integration

Find $\displaystyle\int \frac{3x+1}{(x+1)(x-2)}\,dx$.

Step 1: Decompose: $\dfrac{3x+1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$

Step 2: Multiply through: $3x + 1 = A(x-2) + B(x+1)$

Step 3: Substitute $x = 2$: $7 = 3B \Rightarrow B = \frac{7}{3}$

Step 4: Substitute $x = -1$: $-2 = -3A \Rightarrow A = \frac{2}{3}$

Step 5: Integrate:

$\int \frac{3x+1}{(x+1)(x-2)}\,dx = \frac{2}{3}\ln|x+1| + \frac{7}{3}\ln|x-2| + C$

Q142.

Area Between Two Curves

Find the area enclosed by $y = x^2$ *and* $y = x + 2$.

Step 1: Find intersections: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$

Step 2: On $[-1, 2]$: $x + 2 \geq x^2$ (check at $x = 0$: $2 > 0$). So $f = x + 2, g = x^2$.

Step 3:

$A = \int_{-1}^{2}(x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{10}{3} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$

Q143.

Volume of Revolution

Find the volume generated when $y = \sqrt{x}, 0 \leq x \leq 4$, *is rotated* 360° *about the* x-*axis*.

$V = \pi\int_0^4(\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = \pi \cdot 8 = 8\pi$

Q144.

Kinematics — Finding Position from Acceleration

A particle starts at rest at the origin. Its acceleration is $a(t) = 6t - 4$. Find its position at $t = 3$.

Step 1: $v(t) = \int(6t - 4)\,dt = 3t^2 - 4t + C_1$. Initial rest: $v(0) = 0 \Rightarrow C_1 = 0$.

Step 2: $s(t) = \int(3t^2 - 4t)\,dt = t^3 - 2t^2 + C_2$. Starts at origin: $s(0) = 0 \Rightarrow C_2 = 0$.

Step 3: $s(3) = 27 - 18 = 9$ m

Q145.

Separable DE

Solve $\dfrac{dy}{dx} = xy$, given $y(0) = 2$.

Step 1: Separate: $\dfrac{dy}{y} = x\,dx$

Step 2: Integrate: $\ln|y| = \dfrac{x^2}{2} + C$

Step 3: Exponentiate: $|y| = e^C \cdot e^{x^2/2}$, so $y = Ae^{x^2/2}$ where $A = \pm e^C$

Step 4: Apply $y(0) = 2$: $2 = Ae^0 = A$

Particular solution: $y = 2e^{x^2/2}$

Q146.

Initial Value Problem

Solve $\dfrac{dy}{dx} = \dfrac{x}{y}$, $y(0) = 3$.

Separate: $y\,dy = x\,dx$

Integrate: $\dfrac{y^2}{2} = \dfrac{x^2}{2} + C$

Apply IC: $\dfrac{9}{2} = 0 + C \Rightarrow C = \dfrac{9}{2}$

Particular solution: $y^2 = x^2 + 9$, or $y = \sqrt{x^2 + 9}$ (positive since $y(0) = 3 > 0$)

Q147.

Modelling — Radioactive Decay

A radioactive substance has half-life 8 years. Find: (a) the decay constant k, (b) the fraction remaining after 20 years.

Part (a): At $t = 8$, $N = \frac{1}{2}N_0$:

$\frac{N_0}{2} = N_0 e^{-8k} \Rightarrow \frac{1}{2} = e^{-8k} \Rightarrow -8k = \ln\frac{1}{2} = -\ln 2 \Rightarrow k = \frac{\ln 2}{8}$

Part (b):

$\frac{N}{N_0} = e^{-20k} = e^{-20\ln 2/8} = e^{-5\ln 2/2} = 2^{-5/2} = \frac{1}{4\sqrt{2}} \approx 0.177$

About 17.7% remains after 20 years.

Q148.

Deriving the Maclaurin Series for e^x

Let $f(x) = e^x$. Since $\frac{d}{dx}(e^x) = e^x$, all derivatives equal e^x.

At $x = 0$: $f^{(n)}(0) = e^0 = 1$ for all n.

Substituting into the Maclaurin formula:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Q149.

Deriving the Maclaurin Series for $\sin x$

$f(x) = \sin x$. Compute derivatives at $x = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0

The pattern is $0, 1, 0, -1, 0, 1, \ldots$ Only odd powers survive:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Q150.

Deriving the Maclaurin Series for $\cos x$

$f(x) = \cos x$. Compute derivatives at $x = 0$:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1

The pattern is $1, 0, -1, 0, 1, \ldots$ Only even powers survive, alternating in sign:

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Q151.

Limit Using Maclaurin Series

Evaluate $\lim_{x\to 0} \dfrac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$.

Substitute the series for e^x:

$e^x - 1 - x - \frac{x^2}{2} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right) - 1 - x - \frac{x^2}{2} = \frac{x^3}{6} + \cdots$

$\lim_{x\to 0} \frac{x^3/6+\cdots}{x^3} = \frac{1}{6}$

Q152.

Integration Using Maclaurin Series

Find $\displaystyle\int_0^{0.1} e^{-x^2}\, dx$ *to 5 decimal places.*

Using $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \cdots$:

$\int_0^{0.1} e^{-x^2}\, dx \approx \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2}\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^{0.1}$

$= 0.1 - \frac{0.001}{3} + \frac{0.00001}{10} \approx 0.1 - 0.000333 + 0.000001 = 0.099668$

Q153.

Maclaurin Series Multiplication

Find the Maclaurin series for $\sin x \cdot e^x$ *up to and including the* x^3 *term.*

$\sin x \approx x - \frac{x^3}{6} + \cdots \qquad e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

Multiply term by term, collecting up to x^3:

$x^1 :\ x \cdot 1 = x\ x^2 :\ x \cdot x = x^2\ x^3 :\ x \cdot \frac{x^2}{2} + \left(-\frac{x^3}{6}\right) \cdot 1 = \frac{x^3}{2} - \frac{x^3}{6} = \frac{x^3}{3}$

$\sin x \cdot e^x \approx x + x^2 + \frac{x^3}{3} + \cdots$

Q154.

Q1. The derivative of $f(x) = \ln(\sin x)$ is:

(A) $\cos x$ **(B)** $\cot x$ **(C)** $\tan x$ **(D)** $\dfrac{1}{\sin x}$

Answer: **(B)** Chain rule: $f'(x) = \dfrac{1}{\sin x} \cdot \cos x = \dfrac{\cos x}{\sin x} = \cot x$.

Q155.

Q2. $\displaystyle\int_0^{\pi/2} \sin^2 x\, dx =$

(A) 0 **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{2}$ **(D)** 1

Answer: (B) Use $\sin^2 x = \frac{1-\cos 2x}{2}$:

$\int_0^{\pi/2} \frac{1-\cos 2x}{2}\, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi/2} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Q156.

Q3. If $y = x^x$, then $\frac{dy}{dx} =$

(A) $x \cdot x^{x-1}$ **(B)** $x^x \ln x$ **(C)** $x^x(1 + \ln x)$ **(D)** $x^x \cdot x$

Answer: (C) Use logarithmic differentiation: $\ln y = x \ln x$.

$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$\frac{dy}{dx} = y(\ln x + 1) = x^x(1 + \ln x)$

Q157.

Q4. The value of $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ is:

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1

Answer: (C) Using the Maclaurin series $\cos x = 1 - \frac{x^2}{2} + \cdots$:

$\frac{1-\cos x}{x^2} = \frac{x^2/2-\cdots}{x^2} \to \frac{1}{2}$

Alternatively, apply L'Hôpital's rule twice: $\frac{\sin x}{2x} \to \frac{\cos x}{2} \to \frac{1}{2}$.

Q158.

Q5. The function $f(x) = x^3 - 6x^2 + 9x$ has an inflection point at:

(A) $x = 1$ **(B)** $x = 2$ **(C)** $x = 3$ **(D)** $x = 0$

Answer: (B) $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12 = 6(x-2)$.

$f''(x) = 0$ at $x = 2$. Sign change: $f''(1) = -6 < 0$ and $f''(3) = 6 > 0$ — confirmed sign change from negative to positive. Therefore $x = 2$ is the only inflection point.

Q159.

Q6. $\int x \ln x\, dx =$

(A) $\frac{x^2}{2}\ln x - \frac{x^2}{4} + C$ **(B)** $\frac{x^2 \ln x}{2} + C$ **(C)** $x \ln x - x + C$ **(D)** $\frac{x^2}{4}(\ln x - 1) + C$

Answer: (A) IBP with $u = \ln x, dv = x\,dx$:

$\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

Q160.

Q7. The general solution of $\dfrac{dy}{dx} = 2y$ is:

(A) $y = Ce^x$ **(B)** $y = Ce^{2x}$ **(C)** $y = 2Ce^x$ **(D)** $y = x^2 + C$

Answer: (B) Separate: $\dfrac{dy}{y} = 2\,dx \Rightarrow \ln|y| = 2x + k \Rightarrow y = Ce^{2x}$.

Q161.

Q8. Which of the following is the correct Maclaurin expansion of $\cos(2x)$ up to the x^4 term?

(A) $1 - 2x^2 + \dfrac{2x^4}{3}$ **(B)** $1 - 2x^2 + \dfrac{x^4}{3}$ **(C)** $1 - x^2 + \dfrac{x^4}{6}$ **(D)** $1 - 4x^2 + \dfrac{8x^4}{3}$

Answer: (A) Substitute $2x$ into the standard $\cos x$ series:

$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} = 1 - \frac{4x^2}{2} + \frac{16x^4}{24} = 1 - 2x^2 + \frac{2x^4}{3}$

Common error: substituting $x \to 2x$ but forgetting to apply the exponents — writing $\frac{2x^2}{2}$ instead of $\frac{(2x)^2}{2} = \frac{4x^2}{2}$.

Q162.

Q9. If $f''(x) > 0$ on (a, b) then on (a, b), f is:

(A) decreasing **(B)** increasing **(C)** concave down **(D)** concave up

Answer: (D) $f'' > 0$ means the gradient f' is increasing, which is the definition of concave up. The sign of f'' says nothing directly about whether f is increasing or decreasing (that depends on f').

Q163.

Q10. A particle moves with $v(t) = t^2 - 3t$. The particle first comes to rest at $t =$

(A) 0 **(B)** 1 **(C)** 3 **(D)** 1.5

Answer: (C) $v(t) = 0 \Rightarrow t(t-3) = 0 \Rightarrow t = 0$ or $t = 3$. At $t = 0$ the particle starts from rest, so the **first** time it **comes** to rest again is at $t = 3$.

Q164.

Q11. The area enclosed by $y = e^x$ and the lines $x = 0, x = 1, y = 0$ is:

(A) e **(B)** $e - 1$ **(C)** $e + 1$ **(D)** 1

Answer: (B) $\int_0^1 e^x\,dx = [e^x]_0^1 = e^1 - e^0 = e - 1.$

Q165.

Q12. The volume generated by rotating $y = 2x, 0 \le x \le 3$, about the x-axis is:

(A) 12π **(B)** 24π **(C)** 36π **(D)** 72π

Answer: (C) $V = \pi \int_0^3 (2x)^2\,dx = \pi \int_0^3 4x^2\,dx = 4\pi \left[\frac{x^3}{3}\right]_0^3 = 4\pi \cdot 9 = 36\pi.$

Q166.

Q13. The partial fraction decomposition of $\dfrac{5}{(x-1)(x+4)}$ is:

(A) $\dfrac{1}{x-1} - \dfrac{1}{x+4}$ **(B)** $\dfrac{1}{x-1} + \dfrac{1}{x+4}$ **(C)** $\dfrac{5}{x-1} - \dfrac{5}{x+4}$ **(D)** $\dfrac{1}{x-1} + \dfrac{4}{x+4}$

Answer: (A) Let $\dfrac{5}{(x-1)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x+4}$.

$5 = A(x+4) + B(x-1)$. Set $x = 1$: $5 = 5A \Rightarrow A = 1$. Set $x = -4$: $5 = -5B \Rightarrow B = -1$.

Q167.

Q14. If $\int_0^a x^2\,dx = 9$, then $a =$

(A) 3 **(B)** $\sqrt{3}$ **(C)** $\sqrt[3]{27}$ **(D)** 9

Answer: (A) $\left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} = 9 \Rightarrow a^3 = 27 \Rightarrow a = 3.$

Note: **(C)** is also equal to 3, since $\sqrt[3]{27} = 3$ — but **(A)** is the standard form.

Q168.

Q15. Using the Maclaurin series for $\ln(1+x)$, the approximate value of $\ln(1.1)$ to 4 decimal places (using up to the x^4 term) is:

(A) 0.0953 **(B)** 0.1000 **(C)** 0.0900 **(D)** 0.0909

Answer: (A) With $x = 0.1$:

$\ln(1.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} = 0.1 - 0.005 + 0.000333 - 0.000025 = 0.095308$

Rounded to 4 d.p.: 0.0953.

Q169.

Q16. For $f(x) = x^3 - 3x$, the local minimum value is:

(A) -2 **(B)** 0 **(C)** 2 **(D)** 3

Answer: (A) $f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x = \pm 1$.

$f''(x) = 6x$. At $x = 1$: $f'' = 6 > 0$ (local min). $f(1) = 1 - 3 = -2$.

Q170.

Q17. Given the implicit equation $x^2y + y^3 = 2x$, find $\dfrac{dy}{dx}$ at the point $(1, 1)$.

(A) $\dfrac{1}{4}$ **(B)** 0 **(C)** $-\dfrac{1}{4}$ **(D)** -1

Answer: (B) Differentiate both sides with respect to x:

$\frac{d}{dx}(x^2y + y^3) = \frac{d}{dx}(2x)$

Using product rule on x^2y and chain rule on y^3:

$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2$

Collect terms with $\dfrac{dy}{dx}$:

$\frac{dy}{dx}(x^2 + 3y^2) = 2 - 2xy$

$\frac{dy}{dx} = \frac{2-2xy}{x^2+3y^2}$

At $(1, 1)$: $\dfrac{dy}{dx} = \dfrac{2 - 2(1)(1)}{1^2 + 3(1)^2} = \dfrac{0}{4} = 0$

Common error: Forgetting to apply the product rule to x^2y, leading to option (C).

Q171.

Q18. A ladder 5 m long leans against a vertical wall. The bottom of the ladder slides away from the wall at 1 m/s. How fast is the top of the ladder sliding down the wall when the bottom is 3 m from the wall?

(A) $-\dfrac{3}{8}$ m/s **(B)** $-\dfrac{1}{2}$ m/s **(C)** $-\dfrac{2}{3}$ m/s **(D)** $-\dfrac{3}{4}$ m/s

Answer: (D) Let x = distance from wall to bottom of ladder, y = height of top of ladder.

By Pythagoras: $x^2 + y^2 = 25$

Given: $\dfrac{dx}{dt} = 1$ m/s. Find: $\dfrac{dy}{dt}$ when $x = 3$.

When $x = 3$: $9 + y^2 = 25 \Rightarrow y = 4$ m.

Differentiate the constraint with respect to time:

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

Substitute $x = 3, y = 4$, and $\dfrac{dx}{dt} = 1$:

$\frac{dy}{dt} = -\frac{3}{4}(1) = -\frac{3}{4}\text{ m/s}$

The negative sign indicates the ladder is sliding down.

Common error: Forgetting the negative sign from differentiation, leading to option (C) with positive value, or incorrectly computing $\dfrac{x}{y}$ as $\dfrac{4}{3}$, leading to option (B).

Q172.

Question 1 [Integration by Parts] [~7 marks]

Evaluate the definite integral

$\int_0^\pi x \sin x \, dx$

showing all working. State the integration technique used.

▶ Show Solution

Q173.

Question 2 [Optimization] [~8 marks]

A closed cylindrical tin has a fixed volume of $V = 250\pi \text{ cm}^3$. The total surface area of the tin is S.

(a) Show that $S = 2\pi r^2 + \dfrac{500\pi}{r}$, where r is the radius of the base in cm.

(b) Find the value of r that minimizes S, and verify it is a minimum.

(c) Find the minimum surface area, giving your answer in exact form.

▶ Show Solution

Q174.

Question 3 [Separable Differential Equation] [~7 marks]

Consider the differential equation

$\frac{dy}{dx} = \frac{2x}{y+1}, \quad y > -1$

(a) Find the general solution, expressing y explicitly in terms of x.

(b) Find the particular solution satisfying $y(0) = 2$.

▶ Show Solution

Q175.

Question 4 [Maclaurin Series] [~7 marks]

(a) Write down the Maclaurin series for $\sin u$ up to and including the term in u^5.

(b) Hence find the first three non-zero terms of the Maclaurin series for $\sin(x^2)$.

(c) State the interval of validity of the series in part (b).

(d) Use the series from part (b) to find an approximation for $\int_0^{0.5} \sin(x^2)\, dx$, giving your answer to 4 significant figures.

▶ Show Solution

Q176.

Question 5 [Area Between Curves] [~8 marks]

The curves $f(x) = x^2 - 2x$ and $g(x) = 4 - x^2$ intersect at two points.

(a) Find the x-coordinates of the two intersection points.

(b) Find the exact area of the region enclosed between the two curves.

▶ Show Solution